Section 2.1. Area between curves.

Examples. Find the area of the region bounded by

1. $y = e^x$, $y = 1 - x$, $x = 2$
2. $x = e^y$, $x + y = 1$, $y = 2$

Exercises. Find the area of the region bounded by the following curves.

1. $y = \sqrt{x}$ and $y = 2\sqrt{x}$ between $x = 0$ and $x = 4$

   Answer. $\displaystyle\int_0^4 \sqrt{x}\, dx = \frac{16}{3}$.

2. $y = x^2$ and $y = 2x + 3$

   Answer. $\displaystyle\int_{-1}^3 (2x + 3 - x^2)\, dx = \frac{50}{3}$.

3. $x = 4 - y^2$ and $x = y^2 - 4$

   Answer. $\displaystyle\int_{-2}^2 (8 - 2y^2)\, dy = \frac{64}{3}$.

4. $y = \sin x$ and $y = \cos x$ (one of infinitely many congruent pieces)

   Answer. $\displaystyle\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = 2\sqrt{2}$.

5. $y = \ln x$, x-axis, y-axis, and $y = 1$

   Answer. $\displaystyle\int_0^1 e^y\, dy = e - 1$.

Section 2.2. Volumes by slicing.

Examples. Find the volume of the solid of revolution formed by revolving the region bounded by the graphs of the following functions and rotated about the given line.

1. $y = x^2$, $x = 1$, $x = 4$, and the x-axis; rotated about the x-axis
2. $y = x^2$, $y = 1/x$, and $x = 4$; rotated about the line $y = -1$

Exercises. Find the volume of the solid of revolution formed by revolving the region bounded by the given curves and rotated about the given line.

1. $y = e^x + 1$, $x = 0$, $x = 1$, and $y = 0$; rotated about the x-axis

   Solution. $\int_0^1 \pi(e^x+1)^2 dx = \pi \int_0^1 (e^{2x}+2e^x+1)dx = \pi \left(\frac{1}{2}e^{2x} + 2e^x + x\Big|_0^1 \right) = \pi \left(\frac{1}{2}e^2 + 2e + 1 - \frac{1}{2} - 2\right) = \pi \left(\frac{1}{2}e^2 + 2e - \frac{3}{2}\right)$.
2. $y = x^2$, $x = 0$, and $y = 6$; rotated about the y-axis

   Solution. $\int_0^6 \pi(\sqrt{y})^2 dy = \pi \int_0^6 y dy = \frac{\pi y^2}{2}\Big|_0^6 = 18\pi$.
3. $y = x + 2$, $y = x + 6$, $x = 0$, and $x = 4$; rotated about the x-axis

   Solution. $\int_0^4 \pi((x+6)^2-(x+2)^2)dx = \pi \int_0^4 (8x+32)dx = \pi(4x^2 + 32x)\Big|_0^4 = \pi(64 + 128) = 192\pi$.
4. $y = x^2$ and $y = x^3$; rotated about the line $y = 2$

   Solution. $\int_0^1 \pi((2-x^3)^2 - (2-x^2)^2)dx = \pi \int_0^1 (-4x^3 + x^6 + 4x^2 - x^4)dx = \pi(-x^4 + \frac{x^7}{7} + \frac{4x^3}{3} - \frac{x^5}{5})\Big|_0^1 = \pi(-1 + \frac{1}{7} + \frac{4}{3} - \frac{1}{5}) = \frac{29}{105}\pi$.

Section 2.3. Volumes by cylindrical shells.

Example. Find the volume of the solid of revolution formed by revolving the region bounded by the given curves and rotated about the given line.

1. $y = x^2$, $y = 0$, and $x = 1$; rotated about the y-axis
2. $y = x^2$, $y = 0$, and $x = 1$; rotated about the line $x = 2$

Exercises. Find the volume of the solid of revolution formed by revolving the region bounded by the given curves and rotated about the given line.

1. $y = \frac{1}{x}$, $x = 1$, $x = 4$, and $y = 0$; rotated about the y-axis

   Solution. $\displaystyle\int_1^4 2\pi x \cdot \frac{1}{x}\, dx = \int_1^4 2\pi\, dx = 2\pi x\Big|_1^4 = 6\pi.$

2. $y = \sqrt{x}$ and $y = x^2$; rotated about the line $x = 2$

   Solution. $\displaystyle\int_0^1 2\pi(2-x)(\sqrt{x}-x^2)\, dx = 2\pi \int_0^1 (2x^{1/2} - 2x^2 - x^{3/2} + x^3)\, dx =$ $\displaystyle 2\pi\left(\frac{4x^{3/2}}{3} - \frac{2x^3}{3} - \frac{2x^{5/2}}{5} + \frac{x^4}{4}\right)\Big|_0^1 = \frac{31\pi}{30}.$

3. $x = \frac{1}{y^2+1}$, $x = 0$, $y = 1$, and $y = 4$; rotated about the x-axis

   Solution. $\displaystyle\int_1^4 \frac{2\pi y}{y^2+1}\, dy = \int_2^{17} \frac{\pi}{u}\, du = \pi \ln|u|\Big|_2^{17} = \pi(\ln 17 - \ln 2)$, where $u = y^2 + 1$, $du = 2y\, dy$.

Section 2.4. Arc Length and Surface Area.

Examples.

1. Find the length of the curve $y = 2x^{3/2}$ from $x = 0$ to $x = 1$.
2. Find the area of the surface obtained by rotating the line segment $y = 2x$ from $x = 0$ to $x = 2$ about the x-axis.

Exercises.

1. Find the area of the surface obtained by rotating the line segment $y = 2x$ from $x = 0$ to $x = 2$ about the y-axis.

   Solution. $\displaystyle\int_0^2 2\pi x\sqrt{1+2^2}\,dx = \sqrt{5}\pi x^2\Big|_0^2 = 4\sqrt{5}\pi.$

   Alternatively, you can integrate with respect to y: $\displaystyle\int_0^4 2\pi\frac{y}{2}\sqrt{1+\left(\frac{1}{2}\right)^2}\,dy.$

   The final answer is the same.

2. Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = -1$ to $x = 1$.

   Solution. $\displaystyle\int_{-1}^1 \sqrt{1+\left(\frac{1}{2}(e^x - e^{-x})\right)^2}\,dx = \int_{-1}^1 \sqrt{1+\frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}}\,dx =$

   $\displaystyle\int_{-1}^1 \sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}\,dx = \int_{-1}^1 \sqrt{\left(\frac{1}{2}(e^x + e^{-x})\right)^2}\,dx =$

   $\displaystyle\int_{-1}^1 \frac{1}{2}(e^x + e^{-x})\,dx = \int_0^1 (e^x + e^{-x})\,dx = (e^x - e^{-x}\Big|_0^1 = e - e^{-1}.$

3. Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$ from $x = 1$ to $x = 4$ about the x-axis.

   Solution. $\displaystyle\int_1^4 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = 2\pi\int_1^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx =$

   $\displaystyle 2\pi\int_1^4 \sqrt{x+\frac{1}{4}}\,dx = 2\pi\int_{5/4}^{17/4} \sqrt{u}\,du = \frac{4}{3}\pi u^{3/2}\Big|_{5/4}^{17/4} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}).$

Section 2.5 Density and mass.

Example.

1. Consider a thin rod oriented on the x-axis over the interval $[0, 4]$. If the density function of the rod can be approximated by $\rho(x) = 1 + \sin(\frac{\pi}{4}x)$, what is (approximately) the mass of the rod?

Exercise.

1. Suppose we have some rods, each oriented on the x-axis over the interval $[0, l]$, where l is the length of the rod. The density function of each rod is $\rho(x) = \frac{x+2}{x+1}$. Graph the density function over the interval $[0, 10]$. Which is greater: the mass of one rod of length 10 or the mass of two rods of length 5?

   Solution. Rewrite the function as $\rho(x) = 1 + \frac{1}{x+1}$ and sketch its graph (starting from $y = \frac{1}{x}$ and using graph transformations). This is a decreasing function. Thus the mass of two rods of length 5 is greater. (The difference in mass is $2\int_0^5 \left(1 + \dfrac{1}{x+1}\right) dx - \int_0^{10} \left(1 + \dfrac{1}{x+1}\right) dx = 2\left(x + \ln(x+1)\right)\Big|_0^5 - \left(x + \ln(x+1)\right)\Big|_0^{10} = 2\ln 6 - \ln 11$.)

Section 2.6. Center of mass.

Examples.

1. Find the center of mass of the following system: $m = 1$ at $x = 0$, $m = 2$ at $x = 1$, and $m = 3$ at $x = 2$.
2. Find the center of mass of the following system in the xy-plane: $m = 2$ at $(1, 1)$ and $m = 8$ at $(2, 3)$.
3. Find the center of mass of the rod positioned along the x-axis over the inteval $[0, 1]$ with density function given by $\rho(x) = 1 + \sqrt{x}$.
4. Find the center of mass of the region bounded by $y = 1 + \sqrt{x}$, $y = 0$, $x = 0$, and $x = 1$.

Exercises.

1. Find the center of mass of the following collection: $m = 5$ at $x = -1$, $m = 2$ at $x = 1$, and $m = 3$ at $x = 3$.

   Solution. $x = \dfrac{5(-1) + 2 \cdot 1 + 3 \cdot 3}{10} = \dfrac{3}{5}$.
2. Set up an expression (involving integrals) needed to find the center of mass of the rod of length π units whose density function is given by $\rho(x) = 1+\sin x$. Identify the integrals in the expressions that you already know how to evaluate.

   Answer. $x = \dfrac{\int_0^\pi x(1 + \sin x)\, dx}{\int_0^\pi (1 + \sin x)\, dx}$. The integral in the denominator is easy to evaluate. For the one in the numerator, we do not yet know how to find an antiderivative of $x \sin x$.
3. Set up expressions (involving integrals) needed to find the center of mass of the region bounded by $y = 1 + \sin x$ and $y = 0$ between $x = 0$ and $x = \pi$. Identify the integrals in the expressions that you already know how to evaluate.

   Answer. The x-coordinate is the same as in the problem above. The y-coordinate is given by $y = \dfrac{\frac{1}{2}\int_0^\pi (1 + \sin x)^2\, dx}{\int_0^\pi (1 + \sin x)\, dx}$. Again, the integral in the denominator is easy to evaluate. In the numerator, we can multiply out (foil) the expression. One of the terms will be $\sin^2 x$ that we do not yet know how to integrate.