Problems and Solutions

Edited by Daniel H. Ullman, Daniel J. Velleman, Stan Wagon & Douglas B. West

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below must be submitted by September 30, 2022. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12321. *Proposed by Mohammadamin Sharifi, Sharif University of Technology, Tehran, Iran.* Let p be a prime number, and let N be the number of perfect squares m such that the least nonnegative remainder of $p \pmod m$ is a perfect square. Prove that N is less than $2p^{1/3}$.

12322. *Proposed by Askar Dzhumadil'daev, Kazakh-British Technical University, Almaty, Kazakhstan.* Given real numbers $x_1, \ldots, x_{2n}$, let A be the skew-symmetric $2n$-by-$2n$ matrix with entries $a_{i,j} = (x_i - x_j)^2$ for $1 \le i < j \le 2n$. Prove

$$\det(A) = 4^{n-1}\Big((x_1 - x_2)(x_2 - x_3)\cdots(x_{2n-1} - x_{2n})(x_{2n} - x_1)\Big)^2.$$

12323. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden, and Andreas Dieckmann, Physikalisches Institut der Universität Bonn, Bonn, Germany.* **(a)** Find integers c_0, c_1, and c_2 such that

$$\sum_{k=0}^{\infty} \frac{k^{11}}{(k!)^3} = \sum_{k=0}^{\infty} \frac{c_0 + c_1 k + c_2 k^2}{(k!)^3}.$$

(b) Prove that for any integers n and b with $1 \le b \le n$, there are integers c_m for $c_0, \ldots, c_{b-1}$ such that

$$\sum_{k=0}^{\infty} \frac{k^n}{(k!)^b} = \sum_{k=0}^{\infty} \left(\frac{1}{(k!)^b} \sum_{m=0}^{b-1} c_m k^m \right).$$

(c) Prove that the integers c_m from part **(b)** are unique.

12324. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let a and b be positive real numbers. Prove

$$\int_0^\infty \frac{1}{\sqrt{ax^4 + 2(2b-a)x^2 + a}}\,dx = \int_0^\infty \frac{1}{\sqrt{bx^4 + 2(2a-b)x^2 + b}}\,dx.$$

http://dx.doi.org/doi.org/10.1080/00029890.2022.2044216

12325. *Proposed by Dong Luu, Hanoi University of Education, Hanoi, Vietnam.* Let $ABCD$ be a quadrilateral with a circumscribed circle ω and an inscribed circle γ. Prove that there are two circles α and β with the following property: For any triangle $\triangle MEF$ with (1) M on ω, (2) E and F on the line AB, and (3) the lines ME and MF tangent to γ, the circumcircle of $\triangle MEF$ is tangent to α and β.

12326. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every fixed $y \in \mathbb{R}$, $f(x+y) - f(x)$ is a polynomial in x. Prove that f is a polynomial function.

12327. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \displaystyle\prod_{i=0}^{k-1} \frac{1-q^{n-i}}{1-q^{k-i}} & \text{if } 1 \le k \le n; \\ 1 & \text{if } k = 0. \end{cases}$$

Prove

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2}$$

for $n \ge 0$.

SOLUTIONS

Non-divisors of Translated Sums of Squares

12200 [2020, 660]. *Proposed by Ibrahim Suat Evren, Denizli, Turkey.* Prove that for every positive integer m, there is a positive integer k such that k does not divide $m + x^2 + y^2$ for any positive integers x and y.

Solution by Peter W. Lindstrom, Saint Anselm College, Manchester, NH. We prove that $4m^2$ has the desired property. Let $k = 4m^2$, and let c be a positive integer, so $ck - m = m(4cm - 1)$. Since $4cm - 1 \equiv -1 \pmod 4$, the prime factorization of $4cm - 1$ must have an odd power of a prime p with $p \equiv -1 \pmod 4$. Also, since m and $4cm - 1$ are relatively prime, p cannot divide m, so the prime factorization of $ck - m$ has p to an odd power.

The "sum of two squares" theorem in number theory states that the prime factorization of a number of the form $x^2 + y^2$ has even exponent for each prime congruent to $-1 \pmod 4$. Hence no integers c, x, and y satisfy $x^2 + y^2 + m = ck$. This makes it impossible for k to divide $x^2 + y^2 + m$ for any integers x and y.

Also solved by R. Boukharfane (Saudi Arabia), R. Chapman (UK), C. Curtis & J. Boswell, S. M. Gagola Jr., N. Hodges (UK), E. J. Ionaşcu, Y. J. Ionin, J. S. Liu, O. P. Lossers (Netherlands), S. Miao (China), C. R. Pranesachar (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), K. Williams (Canada), L. Zhou, FAU Problem Solving Group, and the proposer.

A Large Vector Sum from Probability or Polygons

12202 [2020,752]. *Proposed by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China.* Let V be a finite set of vectors in $\mathbb{R}^2$ such that $\sum_{v \in V} |v| = \pi$. Prove that there exists a subset U of V such that $|\sum_{v \in U} v| \ge 1$.

Solution I by Oliver Geupel, Brühl, Germany. Choose at random a ray h starting from the origin. For $v \in V$, let X_v be the length of the projection of v onto h if the angle between them is acute, and 0 otherwise. The expected value of X_v is

$$E[X_v] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |v|\cos\phi\, d\phi = \frac{|v|}{\pi}.$$

Therefore $E[\sum_{v\in V} X_v] = \sum_{v\in V} E[X_v] = 1$, so there is some ray h such that $\sum_{v\in V} X_v \geq 1$. We can now let $U = \{v \in V : \text{the angle between } h \text{ and } v \text{ is acute}\}$.

Solution II by Elton Bojaxhiu, Tirana, Albania, and Enkel Hysnelaj, Sydney, Australia. Let $V = \{v_1, \ldots, v_n\}$, and define v_{n+1} so that $v_1 + \cdots + v_{n+1} = 0$. For any vector v, let $\theta(v)$ be the angle from the positive x-axis to v, with $0 \leq \theta(v) < 2\pi$, and let $v'_1, \ldots, v'_{n+1}$ be a permutation of $v_1, \ldots, v_{n+1}$ such that $\theta(v'_1) \leq \cdots \leq \theta(v'_{n+1})$. The endpoints of the partial sums $\sum_{i=1}^{r} v'_i$ form the vertices of a (possibly degenerate) convex polygon. Let p and d be the perimeter and diameter of this polygon; it is known that $p < \pi d$. Thus

$$\pi = \sum_{v\in V} |v| \leq \sum_{k=1}^{n+1} |v_k| = p < \pi d,$$

so $d > 1$. The set U can be chosen to be a collection of vectors (not including v_{n+1}) whose sum gives a diameter of the polygon.

Editorial comment. Kevin Byrnes and Nicolás Caro pointed out that this problem appears as exercise 14.9 in J. Michael Steele (2004), *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Cambridge: Cambridge Univ. Press, and also in W. W. Bledsoe (1970), An inequality about complex numbers, this MONTHLY 77, pp. 180–182. If p and d are the perimeter and diameter of a convex m-gon, then the inequality $p < \pi d$ follows from $p \leq 2m\sin(\pi/(2m))d$, proved in H. Sedrakyan and N. Sedrakyan (2017), *Geometric Inequalities: Methods of Proving*, Cham, Switzerland: Springer, p. 379. Radouan Boukharfane and Tom Wilde extended the problem to $\mathbb{R}^n$, where the constant π generalizes to $2\sqrt{\pi}\,\Gamma((n+1)/2)/\Gamma(n/2)$.

Also solved by R. Boukharfane (Saudi Arabia), K. M. Byrnes, N. Caro (Brazil), R. Chapman (UK), R. Frank (Germany), Y. J. Ionin, Y. Jeong (Korea), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, K. Schilling, E. Schmeichel, R. Stong, R. Tauraso (Italy), T. Wilde (UK), and the proposer.

A Family of Sums with Logarithmic Powers

12203 [2020, 752]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Let m be a nonnegative integer, and let μ be the Möbius function on $\mathbb{Z}^+$, defined by setting $\mu(k)$ equal to $(-1)^r$ if k is the product of r distinct primes and equal to 0 if k has a square prime factor. Evaluate

$$\lim_{n\to\infty} \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland. The limit is $m+1$.

For a fixed $j \geq 1$, we show that there is a positive constant c such that

$$\sum_{k=1}^{n} \frac{\mu(k)}{k}(-1)^j \ln^j k = \frac{d^j}{ds^j}\frac{1}{\zeta(s)}\bigg|_{s=1} + O\left(e^{-c\sqrt{\ln n}}\right), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta function. We start with

$$\frac{d^j}{ds^j}\frac{1}{\zeta(s)} - \sum_{k=1}^{n} \frac{\mu(k)}{k^s}(-1)^j \ln^j(k) = \sum_{k=n+1}^{\infty} \frac{\mu(k)}{k^s}(-1)^j \ln^j(k) \tag{2}$$

for $s > 1$, which follows from Dirichlet's expansion of $1/\zeta(s)$. We now show that (2) holds also in the case $s = 1$.

Let $M(k) = \sum_{i=1}^{k} \mu(i)$. The function M is known as the Mertens function. Partial summation yields

$$\sum_{k=n+1}^{\infty} \frac{\mu(k)}{k^s}(-1)^j \ln^j(k) = \sum_{k=n+1}^{\infty} \frac{M(k)}{k^s}(-1)^j \ln^j(k) - \sum_{k=n}^{\infty} \frac{M(k)}{(k+1)^s}(-1)^j \ln^j(k+1)$$

$$= \frac{M(n)}{n^s}(-1)^{j+1} \ln^j(n) + \sum_{k=n}^{\infty} M(k)(-1)^j \left(\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s}\right).$$

For $s \geq 1$ and $x > e^j$,

$$\frac{d}{dx}\frac{\ln^j(x)}{x^s} = \frac{\ln^j(x)}{x^{s+1}}\left(\frac{j}{\ln x} - s\right) < 0.$$

Moreover,

$$\frac{d}{dx}\frac{\ln^j(x)}{x^s} > -s\,\frac{\ln^j(x)}{x^{s+1}},$$

with the latter increasing in x. Thus, by the mean value theorem,

$$\left|\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s}\right| < s\,\frac{\ln^j(k)}{k^{s+1}} \leq 2\,\frac{\ln^j(k)}{k^{s+1}}$$

for $1 \leq s \leq 2$ and $k > e^j$. Since $M(k) = O\left(ke^{-2c\sqrt{\ln k}}\right)$ for a suitable positive constant c (see, for instance, E. Landau (1974), *Handbuch der Lehre von der Verteilung der Primzahlen*, v. 2, AMS Chelsea Publishing: Providence, p. 570) and since $\ln^{j+2}(k) = O\left(e^{c\sqrt{\ln k}}\right)$, we have

$$\left|M(k)(-1)^j \left(\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s}\right)\right| = O\left(e^{-c\sqrt{\ln k}}\,\frac{1}{k\ln^2(k)}\right).$$

From this we deduce

$$\left|\frac{M(n)}{n^s}(-1)^{j+1} \ln^j(n) + \sum_{k=n}^{\infty} M(k)(-1)^j \left(\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s}\right)\right|$$

$$= O\left(e^{-c\sqrt{\ln n}}\right) + \sum_{k=n}^{\infty} O\left(e^{-c\sqrt{\ln k}}\,\frac{1}{k\ln^2(k)}\right)$$

$$= O\left(e^{-c\sqrt{\ln n}}\right) + O\left(e^{-c\sqrt{\ln n}}\frac{1}{\ln n}\right) = O\left(e^{-c\sqrt{\ln n}}\right).$$

The convergence of the series is uniform for $s \in [1, 2]$, so both sides of (2) are continuous on $[1, 2]$. Therefore, (2) is valid at $s = 1$, proving (1).

We conclude

$$\frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right) = \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} (\ln n - \ln k)^{m+1}$$

$$= \frac{1}{\ln^m(n)} \sum_{j=0}^{m+1} \binom{m+1}{j} \ln^{m+1-j}(n) \sum_{k=1}^{n} \frac{\mu(k)}{k}(-1)^j \ln^j(k)$$

$$= \frac{1}{\ln^m(n)} \sum_{j=0}^{m+1} \binom{m+1}{j} \ln^{m+1-j}(n) \left(\left.\frac{d^j}{ds^j}\frac{1}{\zeta(s)}\right|_{s=1} + O\left(e^{-c\sqrt{\ln n}}\right)\right).$$

As $n \to \infty$, all error terms have limit 0. Since $\zeta(s)$ is meromorphic with a simple pole of residue 1 at $s = 1$, the function $1/\zeta(s)$ is holomorphic at $s = 1$, and its Taylor series expansion begins $(s - 1) + \cdots$. The main term vanishes for $j = 0$ and has limit 0 for $j > 1$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right) = \binom{m+1}{1} \frac{d}{ds} \frac{1}{\zeta(s)}\bigg|_{s=1} = m + 1.$$

Editorial comment. The proof of the bound on the Mertens function is similar to one for the prime number theorem. Some solvers used other bounds, shortening their solutions. Bounds on sums of the form $\sum_{k=1}^{n} \mu(k) \ln^q(k)/k$ (Landau, pp. 568–570, 594–595) allow one to begin with the binomial expansion of $\ln n - \ln k$. For $m > 0$, the solution follows immediately from

$$\sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right) = (m+1)\ln^m(n) + \sum_{k=1}^{m-1} c_k(m) \ln^k(n) + O(1),$$

which appears on p. 489 of H. N. Shapiro (1950), On a theorem of Selberg and generalizations, *Ann. Math.*, 485–497.

Also solved by W. Janous (Austria), A. Stenger, R. Stong, and the proposer.

The Sum of Cosines in a Convex Quadrilateral

12204 [2020, 752]. *Proposed by Florentin Visescu, Bucharest, Romania.* Prove that the absolute value of the sum of the cosines of the four angles in a convex quadrilateral is less than 1/2.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Denote the angles by α_i for $i \in \{1, 2, 3, 4\}$, with $0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4 < \pi$. We have $\sum \alpha_i = 2\pi$. Let $a = \alpha_1 + \alpha_2$, and note that $a \le \pi$ and $\alpha_3 + \alpha_4 = 2\pi - a$. If $a = \pi$, then all four angles are $\pi/2$, so $\sum \cos(\alpha_i) = 0$, so $\sum \alpha_i = 0$ and the required inequality holds. We may therefore assume $a < \pi$.

For the sum of the first two cosines,

$$\cos\alpha_1 + \cos\alpha_2 = 2\cos\left(\frac{a}{2}\right)\cos\left(\frac{\alpha_2 - \alpha_1}{2}\right). \tag{1}$$

Since $0 < \alpha_1 \le \alpha_2$, we have

$$0 \le \frac{\alpha_2 - \alpha_1}{2} < \frac{\alpha_1 + \alpha_2}{2} = \frac{a}{2} < \frac{\pi}{2},$$

and therefore

$$\cos\left(\frac{a}{2}\right) < \cos\left(\frac{\alpha_2 - \alpha_1}{2}\right) \le 1.$$

Multiplying by $2\cos(a/2)$, which is positive, we conclude

$$2\cos^2\left(\frac{a}{2}\right) < 2\cos\left(\frac{a}{2}\right)\cos\left(\frac{\alpha_2 - \alpha_1}{2}\right) \le 2\cos\left(\frac{a}{2}\right),$$

which by (1) implies

$$2\cos^2\left(\frac{a}{2}\right) < \cos\alpha_1 + \cos\alpha_2 \le 2\cos\left(\frac{a}{2}\right). \tag{2}$$

Since $0 < \pi - \alpha_4 \le \pi - \alpha_3 < \pi$ and

$$(\pi - \alpha_4) + (\pi - \alpha_3) = 2\pi - (\alpha_3 + \alpha_4) = a,$$

we can apply the same reasoning to $\pi - \alpha_4$ and $\pi - \alpha_3$ to obtain

$$2\cos^2\left(\frac{a}{2}\right) < \cos(\pi - \alpha_4) + \cos(\pi - \alpha_3) \le 2\cos\left(\frac{a}{2}\right),$$

or equivalently

$$-2\cos\left(\frac{a}{2}\right) \le \cos\alpha_3 + \cos\alpha_4 < -2\cos^2\left(\frac{a}{2}\right). \tag{3}$$

Adding (2) and (3), and putting $x = \cos(a/2)$, we get

$$2x^2 - 2x < \sum \alpha_i < 2x - 2x^2.$$

Since the quadratic $2x - 2x^2$ has maximum value 1/2 at $x = 1/2$, this proves the inequality.

Editorial comment. The problem statement assumes that all angles are strictly less than π. If one allows an angle to equal π, then one can achieve a cosine sum of 1/2 by beginning with an equilateral triangle and adding a fourth vertex along one side, obtaining a four-sided figure with angles $\pi/3, \pi/3, \pi/3$, and π. One can obtain quadrilaterals with all angles less than π and cosine sum arbitrarily close to $1/2$ by using angles $\pi/3 + \epsilon$, $\pi/3 + \epsilon$, $\pi/3 + \epsilon$, and $\pi - 3\epsilon$.

Nicolás Caro solved the more general problem of bounding $\sum_{i=1}^{n} \cos x_i$, given that $0 < x_i < \pi$ and $\sum_{i=1}^{n} x_i = j\pi$; the stated problem is the case $n = 4$, $j = 2$.

Also solved by E. Bojazhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (Saudi Arabia), N. Caro (Brazil), R. Chapman (UK), C. Chiser (Romania), G. Fera & G. Tescaro (Italy), L. Giugiuc (Romania), J.-P. Grivaux (France), N. Hodges (UK), E. J. Ionaşcu, Y. J. Ionin, W. Janous (Austria), A. B. Kasturiarachi, O. Kouba (Syria), K.-W. Lau (China), Z. Lin (China), J. H. Lindsey II, K. Park (Korea), C. Schacht, E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, L. Zhou, and the proposer.

Minimizing a Ratio of Integrals

12205 [2020, 752]. *Proposed by Christian Chiser, Elena Cuza College, Craiova, Romania.* Find the minimum value of

$$\frac{\int_0^1 x^2 (f'(x))^2\, dx}{\int_0^1 x^2 (f(x))^2\, dx}$$

over all nonzero continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ with $f(1) = 0$.

Solution by Jinhai Yan, Fudan University, Shanghai, China. We show that the minimum value is π^2.

Let

$$g(x) = \begin{cases} \sin(\pi x)/x, & \text{if } x \neq 0, \\ \pi, & \text{if } x = 0. \end{cases}$$

Note that $g \in C^\infty[0, 1]$, $g(1) = 0$, and g satisfies the Euler–Lagrange equation

$$\frac{d}{dx}\left(x^2 g'(x)\right) = -\pi^2 x^2 g(x).$$

Therefore, for any f as in the problem statement,

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2 g'(x)}{g(x)} f(x)^2\right) &= x^2\left(\frac{2g'(x)}{g(x)} f(x) f'(x) - \pi^2 f(x)^2 - \frac{g'(x)^2}{g(x)^2} f(x)^2\right) \\ &= x^2\left(f'(x)^2 - \pi^2 f(x)^2\right) - x^2\left(f'(x) - \frac{g'(x)}{g(x)} f(x)\right)^2. \end{aligned}$$

Note that the singularity at $x = 1$ on both sides of this equation is removable, because

$$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} \frac{f'(x)}{g'(x)} = -\frac{f'(1)}{\pi} \in \mathbb{R}.$$

It follows that

$$\int_0^1 \left(x^2 \left(f'(x)^2 - \pi^2 f(x)^2 \right) - x^2 \left(f'(x) - \frac{g'(x)}{g(x)} f(x) \right)^2 \right) dx = \left. \frac{x^2 g'(x)}{g(x)} f(x)^2 \right|_0^1 = 0.$$

Thus

$$\int_0^1 x^2 f'(x)^2 \, dx - \pi^2 \int_0^1 x^2 f(x)^2 \, dx = \int_0^1 x^2 \left(f'(x) - \frac{g'(x) f(x)}{g(x)} \right)^2 dx \geq 0,$$

with equality if $f = g$, and the desired conclusion follows.

Also solved by K. F. Andersen (Canada), R. Boukharfane (Saudi Arabia), P. Bracken, H. Chen, T. Dickens, L. Han, O. Kouba (Syria), P. W. Lindstrom, A. Natian, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, and the proposer.

A Skew-Harmonic Formula for Apéry's Constant

12206 [2020, 752]. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n^2} = \frac{3}{4}\zeta(3),$$

where $\overline{H}_n$ is the nth *skew-harmonic number* $\sum_{k=1}^{n} (-1)^{k+1}/k$ and $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$.

Solution by Michel Bataille, Rouen, France. With $H_0 = 0$ and $H_n = \sum_{k=1}^{n} 1/k$,

$$\overline{H}_{2m} = H_{2m} - 2\sum_{k=1}^{m} \frac{1}{2k} = H_{2m} - H_m = \sum_{k=1}^{m} \frac{1}{m+k}. \tag{1}$$

Also note that

$$H_{2m-1} - H_{m-1} - \sum_{j=m}^{m+N} \left(\frac{1}{j} - \frac{1}{j+m} \right) = H_{2m+N} - H_{m+N} = \sum_{j=m+N+1}^{2m+N} \frac{1}{j}.$$

As N tends to ∞, the right side tends to 0, so

$$\sum_{j=m}^{\infty} \left(\frac{1}{j} - \frac{1}{j+m} \right) = H_{2m-1} - H_{m-1}. \tag{2}$$

Let $S = \sum_{n=1}^{\infty} \overline{H}_{2n}/n^2$. By (1),

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{n+k} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} \left(\frac{1}{n} - \frac{1}{n+k} \right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{nk(n+k)}. \tag{3}$$

We consider the two terms in this expression separately. First

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \sum_{n=1}^{\infty} \left(\frac{H_{n-1}}{n^2} + \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} + \zeta(3) = 2\zeta(3)$$

by Euler's formula $\sum_{n=1}^{\infty} H_{n-1}/n^2 = \zeta(3)$.

To evaluate the double sum in the second term of (3), interchange the order of summation, use (2), and then manipulate the harmonic terms and use the first part of (1) to obtain

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{nk(n+k)} = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+k} \right) = \sum_{k=1}^{\infty} \frac{H_{2k-1} - H_{k-1}}{k^2}$$

$$= \sum_{k=1}^{\infty} \frac{H_{2k} - H_k + 1/(2k)}{k^2} = \sum_{k=1}^{\infty} \frac{\overline{H}_{2k}}{k^2} + \frac{\zeta(3)}{2} = S + \frac{\zeta(3)}{2}.$$

Thus

$$S = 2\zeta(3) - \left(S + \frac{\zeta(3)}{2} \right),$$

and the result follows.

Editorial comment. A simple proof of Euler's formula for $\zeta(3)$ appears in this MONTHLY 127 (2020), 855. That issue contains the solutions to Problem 12091 and Problem 12102, both of which also link $\zeta(3)$ to infinite series involving harmonic sums.

Many solvers expressed harmonic numbers as integrals from 0 to 1 of the formula for the sum of a finite geometric series and then performed interchanges. This led to various integrals with logarithmic integrands and/or dilogarithms. Two known definite integrals that played a role in many solutions were

$$\int_1^1 \frac{\log^2(1-x)}{x}\, dx = 2\zeta(3)$$

and

$$\int_0^1 \frac{\log(1-x)\log(1+x)}{x}\, dx = -\frac{5}{8}\zeta(3).$$

Also solved by A. Berkane (Algeria), N. Bhandari (Nepal), R. Boukharfane (Saudi Arabia), K. N. Boyadzhiev, P. Bracken, B. Bradie, N. Caro (Brazil), A. C. Castrillón (Colombia), H. Chen, N. S. Dasireddy (India), G. Fera (Italy), M. L. Glasser, R. Gordon, H. Grandmontagne (France), L. Han, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), I. Mezö (China), R. Molinari, V. H. Moll & T. Amdeberhan, K. Nelson, M. Omarjee (France), S. Sharma (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Wangshinghin (Canada), T. Wiandt, Y. Xiang (China), and the proposer.

A Fibonacci Inequality

12213 [2020, 853]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove

$$\sum_{k=1}^{n} \sqrt{F_{k-1}F_{k+2}} \leq \sqrt{F_{n+1}F_{n+4}} - \sqrt{5}.$$

Solution by Rory Molinari, Beverly Hills, MI. More generally, consider a sequence $\langle a \rangle$ of nonnegative real numbers such that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. For $n \geq 2$ and d a nonnegative integer, we prove

$$\sum_{k=1}^{n-1} \sqrt{a_{k-1}a_{k+d-1}} \leq \sqrt{a_n a_{n+d}} - \sqrt{a_1 a_{d+1}}.$$

Setting $a_n = F_{n+1}$ and $d = 3$ proves the desired inequality.

The identity $\sum_{k=j}^{m} a_k = a_{m+2} - a_{j+1}$ is easily shown by induction on m. By the Cauchy–Schwarz inequality,

$$\sum_{k=1}^{n-1} \sqrt{a_{k-1}a_{k+d-1}} \leq \left(\sum_{k=1}^{n-1} a_{k-1}\right)^{1/2} \left(\sum_{k=1}^{n-1} a_{k+d-1}\right)^{1/2} = \sqrt{(a_n - a_1)(a_{n+d} - a_{d+1})}.$$

By the AM-GM inequality,

$$\begin{aligned}(a_n - a_1)(a_{n+d} - a_{d+1}) &= a_n a_{n+d} + a_1 a_{d+1} - a_1 a_{n+d} - a_{d+1} a_n \\ &\leq a_n a_{n+d} + a_1 a_{d+1} - 2\sqrt{a_1 a_{n+d} a_{d+1} a_n} \\ &= \left(\sqrt{a_n a_{n+d}} - \sqrt{a_1 a_{d+1}}\right)^2.\end{aligned}$$

Editorial comment. The majority of solvers proved the inequality by induction, showing

$$\sqrt{F_{n+1}F_{n+4}} + \sqrt{F_n F_{n+3}} \leq \sqrt{F_{n+2}F_{n+5}}$$

by squaring both sides and applying the AM-GM inequality. Doyle Henderson used this approach to generalize to a sequence of real numbers satisfying $a_n \geq a_{n-1} + a_{n-2}$ for $n \geq 2$ and $\sqrt{a_0 a_3} \leq \sqrt{a_2 a_5} - \sqrt{a_5}$, obtaining

$$\sum_{k=1}^{n} \sqrt{a_{k-1}a_{k+2}} \leq \sqrt{a_{n+1}a_{n+4}} - \sqrt{a_5}.$$

Also solved by K. F. Andersen (Canada), M. Bataille (France), B. D. Beasley, R. Boukharfane (Saudi Arabia), P. Bracken, B. Bradie, Ó. Ciaurri (Spain), C. Curtis, A. Dixit (India) & S. Pathak (USA), G. Fera (Italy), D. Fleischman, O. Geupel (Germany), R. Gordon, D. Henderson, N. Hodges (UK), Y. J. Ionin, W. Janous (Austria), M. Kaplan & M. Goldenberg, K. T. L. Koo (China), O. Kouba (Syria), W.-K. Lai, P. Lalonde (Canada), K.-W. Lau (China), O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), J. Pak (Canada), A. Pathak (India), Á. Plaza (Spain), E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. B. Tyler, J. Van hamme (Belgium), M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, L. Wimmer (Germany), X. Ye (China), A. Zaidan, L. Zhou, FAU Problem Solving Group, and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

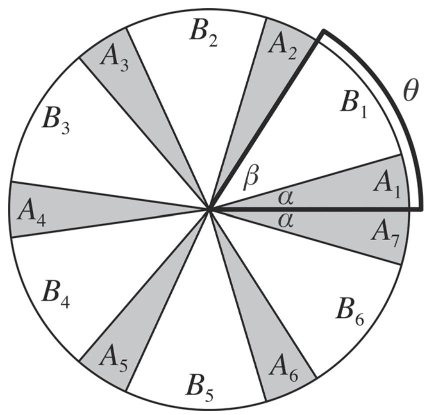
C5. *Due to Victor Klee, contributed by the editors*. Given a set S in $\mathbb{R}^n$, let $L(S)$ be the set of all points lying on some line determined by two points in S. For example, if S is the set of vertices of an equilateral triangle in $\mathbb{R}^2$, then $L(S)$ is the union of the three lines that extend the sides of the triangle, and $L(L(S))$ is all of $\mathbb{R}^2$. If S is the set of vertices of a regular tetrahedron, then what is $L(L(S))$?

Returning the Icing to the Top

C4. *From the 1968 Moscow Mathematical Olympiad, contributed by the editors*. A round cake has icing on the top but not the bottom. Cut a piece of the cake in the usual shape of a sector with vertex angle one radian and with vertex at the center of the cake. Remove the piece, turn it upside down, and replace it in the cake to restore roundness. Next, move one radian around the cake, cut another piece with the same vertex angle adjacent to the first, remove it, turn it over, and replace it. Keep doing this, moving around the cake one radian at a time, inverting each piece. Show that, after a finite number of steps, all the icing will again be on the top.

Solution. We solve the general problem in which the central angle of every slice is θ radians. If $2\pi/\theta$ is an integer n, then clearly n flips put all the icing on the bottom, and n more flips return it all to the top. Otherwise, let $n = \lfloor 2\pi/\theta \rfloor$. We show that the icing returns to the top for the first time after $2n(n+1)$ steps. In the case $\theta = 1$, we have $n = 6$, and therefore it takes 84 steps for the icing to return to the top.

Let $\alpha = 2\pi - n\theta$. Clearly $0 < \alpha < \theta$. Let $\beta = \theta - \alpha$, so that $\alpha + \beta = \theta$. Cut n consecutive pieces with angle θ (these are the first n pieces to be flipped), leaving a piece with angle α. Cut each of the n pieces into two pieces of angle α and β, as in the figure. Reading counterclockwise, you now have pieces of width $\alpha, \beta, \alpha, \beta, \ldots, \alpha$, with the last α adjacent to the first. Let $A_1, \ldots, A_{n+1}$ be the pieces with angle α, and let $B_1, \ldots, B_n$ be the pieces with angle β, with B_i between A_i and A_{i+1}, as shown here. You may now discard the knife; no further cutting is necessary.



Imagine that the cake is on a rotating cake plate and we rotate the cake plate clockwise through an angle of θ after each piece is flipped. In the first step, we flip the piece consisting of A_1 and B_1 and then rotate the plate clockwise. Piece A_1 is now upside down in the original location of piece A_{n+1}, and B_1 is now upside down in the original location of piece B_n. All other pieces simply rotate clockwise without being flipped, so for $2 \le i \le n+1$, A_i moves to the original location of A_{i-1}, and for $2 \le i \le n$, B_i moves to the original location of B_{i-1}. At the end of this operation the cuts are in the same positions as they were in originally; the net effect of one step is simply to permute the A and B pieces cyclically, with one of each being flipped.

It is now clear that after n steps the B pieces have completed a full rotation, with each piece being flipped once, so they are back in their original positions upside down, and after another n steps they are in their original positions right side up again. Similarly, it takes $2(n+1)$ steps for all the A pieces to return to right side up, in their original positions. It follows that the number of steps needed to return all icing to the top is the least common multiple of $2n$ and $2(n+1)$, which is $2n(n+1)$. Indeed, after this many steps, not only is the icing on top, but the cake is fully restored to its original configuration.

Editorial comment. This problem appeared, in a somewhat different form, as problem 31.2.8.3 in the 1968 Moscow Mathematical Olympiad. The version given here appears in P. Winkler (2007), *Mathematical Mind-Benders*, A K Peters/CRC Press, Wellesley, MA.