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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at americanmathematicalmonthly.submittable.com/submit. *Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by March 31, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

12349. *Proposed by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* Let A_n be the set of permutations of $\{1, \ldots, n\}$ that have at least one fixed point. For $\pi \in A_n$, we write $\mathrm{Fix}(\pi)$ for $\{j : \pi(j) = j\}$. Evaluate

$$\sum_{\pi \in A_n} \left(\frac{\mathrm{sign}(\pi)}{|\mathrm{Fix}(\pi)|} \sum_{j \in \mathrm{Fix}(\pi)} j \right).$$

12350. *Proposed by Nick MacKinnon, Stanbury, UK.* What is the smallest positive integer k such that for any quadratic polynomial P with integer coefficients, one of the integers $P(1), \ldots, P(k)$ has a zero digit when written in base two?

12351. *Proposed by Seán Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.* Evaluate

$$\int_0^\infty \frac{\ln\left(\cos^2 x\right) \sin^3 x}{x^3 \left(1 + 2\cos^2 x\right)}\, dx.$$

12352. *Proposed by Haoran Chen, Xi'an Jiaotong–Liverpool University, Suzhou, China.*
(a) Suppose G is a bipartite planar graph such that for any two vertices A and B, the number of shortest paths from A to B is odd. Prove that G is a tree.
(b)* Suppose G is a bipartite planar graph such that for any two vertices A and B, the number of paths from A to B is odd. Must G be a tree?

12353. *Proposed by Yongge Tian, Shanghai Business School, Shanghai, China.* Let A be a square matrix with complex entries, and let A^* denote the conjugate transpose of A. Show that $AA^* = A^*A$ if and only if $\mathrm{rank}\left(A^2\right) = \mathrm{rank}(A)$, $A^2A^* = A^*A^2$, and $A^3A^* = A^*A^3$.

12354. *Proposed by Slobodan Filipovski, University of Primorska, Koper, Slovenia.* Let n and k be positive integers with $n \geq 3$, and let $p(x) = x^n + x^{n-1} + \cdots + x - k$.

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(a) Prove that the roots of $p(x)$ in the complex plane are simple.
(b) Prove that if $k \geq n+1$, then $p(x)$ has at least one root with negative real part and nonzero imaginary part.

12355. *Proposed by Cezar Alexandru Trancanau and Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let x, y, and z be the side lengths of a nondegenerate, nonequilateral triangle with largest angle α. Let T be the set of lengths t such that there exists an equilateral triangle ABC in the plane with origin O such that $AB = t$, $OA = x$, $OB = y$, and $OC = z$.
(a) Prove that $|T| = 2$.
(b) Prove that the smaller of the two equilateral triangles determined by T does not contain O in its interior.
(c) Prove that the larger of the two equilateral triangles determined by T contains O in its interior if and only if $\pi/3 < \alpha < 2\pi/3$.

SOLUTIONS

Making Equality Improbable with Two Dice

12223 [2021, 88]. *Proposed by Michael Elgersma, Plymouth, MN, and James R. Roche, Ellicott City, MD.* Two weighted m-sided dice have faces labeled with the integers 1 to m. The first die shows the integer i with probability p_i, while the second die shows the integer i with probability r_i. Alice rolls the two dice and sums the resulting integers; Bob then independently does the same.
(a) For each m with $m \geq 2$, find the probability vectors $(p_1, \ldots, p_m)$ and $(r_1, \ldots, r_m)$ that minimize the probability that Alice's sum equals Bob's sum.
(b)* Generalize to n dice, with $n \geq 3$.

Composite solution to part **(a)** *by the proposers and Shuyang Gao, George Washington University, Washington, DC.* The minimum probability is $3/(6m-4)$, achieved only by the two distributions

$$\left(\frac{1}{2}, 0, 0, \ldots, 0, 0, \frac{1}{2}\right) \qquad \text{and} \qquad \frac{1}{3m-2}(2, 3, 3, \ldots, 3, 3, 2).$$

We start with some notation. We write $\mathbf{v}$ for a probability (row) vector $(v_1, \ldots, v_m)$ associated with the faces of an m-sided die; that is, the probability that a toss of such a die turns up value i is v_i (similarly with other letters). The reverse $R(\mathbf{v})$ of $\mathbf{v}$ is $(v_m, \ldots, v_1)$. We say that $\mathbf{v}$ is *symmetric* if $\mathbf{v} = R(\mathbf{v})$. For symmetrization and antisymmetrization, let $S_{\mathbf{v}} = (\mathbf{v} + R(\mathbf{v}))/2$ and $A_{\mathbf{v}} = (\mathbf{v} - R(\mathbf{v}))/2$. Thus $\mathbf{v} = S_{\mathbf{v}} + A_{\mathbf{v}}$, $R(S_{\mathbf{v}}) = S_{\mathbf{v}}$, and $R(A_{\mathbf{v}}) = -A_{\mathbf{v}}$.

Let $\mathbf{p}$ and $\mathbf{r}$ denote the probability vectors for the two dice. Let X and Y be the sums rolled by Alice and Bob, respectively. Note that X and Y have the same distribution. Let $\mathbf{s} = (s_2, \ldots, s_{2m})$, where

$$s_k = \mathbb{P}(X = k) = \mathbb{P}(Y = k) = \sum_{i=1}^{m} p_i r_{k-i},$$

with the understanding that $r_j = 0$ unless $1 \leq j \leq m$. With $*$ denoting convolution of vectors, we write $\mathbf{s}$ as $\mathbf{p} * \mathbf{r}$.

Our first task is to show that the probability is minimized only when $\mathbf{p}$ and $\mathbf{r}$ are symmetric. The tool for this is the claim

$$\mathbb{P}(X = Y) \geq (S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}}),$$

with equality holding if and only if $\mathbf{p}$ and $\mathbf{r}$ are both symmetric probability vectors. Given this, let $\mathbf{p}$ and $\mathbf{r}$ be minimizing probability vectors. If we replace $\mathbf{p}$ and $\mathbf{r}$ by their symmetrizations $S_{\mathbf{p}}$ and $S_{\mathbf{r}}$, then the new resulting probability $\mathbb{P}(X = Y)$ will be equal to $(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}})$, which will be strictly smaller than the original probability unless $\mathbf{p} = S_{\mathbf{p}}$ and $\mathbf{r} = S_{\mathbf{r}}$.

Hence we proceed to the claim. Since the players' rolls are independent,

$$\mathbb{P}(X = Y) = \sum_{k=2}^{2m} \mathbb{P}(X = k)\,\mathbb{P}(Y = k) = \sum_{k=2}^{2m} \Big(\sum_{i=1}^{m} p_i r_{k-i} \Big)^2.$$

We write this using convolution and inner product as

$$\mathbb{P}(X = Y) = (\mathbf{p} * \mathbf{r}) \cdot (\mathbf{p} * \mathbf{r}) = \big((S_{\mathbf{p}}+A_{\mathbf{p}}) * (S_{\mathbf{r}}+A_{\mathbf{r}})\big) \cdot \big((S_{\mathbf{p}}+A_{\mathbf{p}}) * (S_{\mathbf{r}}+A_{\mathbf{r}})\big).$$

By linearity of convolution and inner product, this expression expands into sixteen terms of the form $(f_{\mathbf{p}} * g_{\mathbf{r}}) \cdot (h_{\mathbf{p}} * i_{\mathbf{r}})$ with $f, g, h, i \in \{S, A\}$. We show that the contribution from the terms other than $(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}})$ is nonnegative and is 0 if and only if $\mathbf{p}$ and $\mathbf{r}$ are symmetric.

Since $S_{\mathbf{p}} * S_{\mathbf{r}}$ and $A_{\mathbf{p}} * A_{\mathbf{r}}$ are symmetric and $S_{\mathbf{p}} * A_{\mathbf{r}}$ and $A_{\mathbf{p}} * S_{\mathbf{r}}$ are antisymmetric, each of the eight terms having one or three factors in $\{A_{\mathbf{p}}, A_{\mathbf{r}}\}$ is the dot product of a symmetric and an antisymmetric vector and hence vanishes.

With $f, g \in \{S, A\}$, we find four terms of the form $(f_{\mathbf{p}} * g_{\mathbf{r}}) \cdot (f_{\mathbf{p}} * g_{\mathbf{r}})$. Each is nonnegative, since it is the dot product of a vector with itself, and it equals 0 if and only if $f_{\mathbf{p}} * g_{\mathbf{r}} = \mathbf{0}$. The convolution is $\mathbf{0}$ when $f = A$ and $\mathbf{p}$ is symmetric, since then $A_{\mathbf{p}} = \mathbf{0}$. However, if $\mathbf{p}$ is not symmetric, then $A_{\mathbf{p}} * S_{\mathbf{r}} \neq \mathbf{0}$. The corresponding statements hold also for g. Hence the contribution from these four terms is at least $(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}})$, with equality if and only if both $\mathbf{p}$ and $\mathbf{r}$ are symmetric.

The remaining four terms use each factor in $\{S_{\mathbf{p}}, S_{\mathbf{r}}, A_{\mathbf{p}}, A_{\mathbf{r}}\}$. They sum to

$$2\big((S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * A_{\mathbf{r}}) + (S_{\mathbf{p}} * A_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * S_{\mathbf{r}})\big). \tag{1}$$

We claim that this sum is 0. We have

$$(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * A_{\mathbf{r}}) = \sum S_{\mathbf{p}}(k) S_{\mathbf{r}}(\ell) A_{\mathbf{p}}(k') A_{\mathbf{r}}(\ell') \tag{2}$$

and

$$(S_{\mathbf{p}} * A_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * S_{\mathbf{r}}) = \sum S_{\mathbf{p}}(k) A_{\mathbf{r}}(\ell') A_{\mathbf{p}}(k') S_{\mathbf{r}}(\ell), \tag{3}$$

where the sum in (2) is over choices of k, ℓ, k', ℓ' in $\{1, \ldots, m\}$ such that $k + \ell = k' + \ell'$, and the sum in (3) is over choices such that $k + \ell' = k' + \ell$. Note that $k + \ell = k' + \ell'$ if and only if $k - k' = \ell' - \ell$ and that $k + \ell' = k' + \ell$ if and only if $k - k' = \ell - \ell'$. By symmetry and antisymmetry,

$$S_{\mathbf{r}}(\ell) = S_{\mathbf{r}}(m - \ell + 1) \qquad \text{and} \qquad A_{\mathbf{r}}(\ell') = -A_{\mathbf{r}}(m - \ell' + 1).$$

Thus $S_{\mathbf{p}}(k) S_{\mathbf{r}}(\ell) A_{\mathbf{p}}(k') A_{\mathbf{r}}(\ell') = -S_{\mathbf{p}}(k) S_{\mathbf{r}}(m - \ell + 1) A_{\mathbf{p}}(k') A_{\mathbf{r}}(m - \ell' + 1)$. When we require $k - k' = \ell' - \ell$, at the same time we have $k - k' = (m - \ell + 1) - (m - \ell' + 1)$. Hence terms in the sum in (3) negate corresponding terms in the sum in (2), and the expression in (1) is 0. This completes the proof of the claim.

The claim implies the desired result in the case $m = 2$, giving $\mathbf{p} = \mathbf{r} = (1/2, 1/2)$. For the remainder of the argument, we assume $m \geq 3$. With $\mathbf{p}$ and $\mathbf{r}$ symmetric, the convolution $\mathbf{s}$ is also a symmetric probability vector, and the desired probability is $\sum_{k=2}^{2m} s_k^2$. By symmetry,

$$s_{m+1} = \sum_{i=1}^{m} p_i r_{m-i+1} \geq p_1 r_m + p_m r_1 = 2 p_1 r_1 = 2 s_2. \tag{4}$$

This suggests that we consider the following nonlinear optimization problem:

$$\text{minimize} \qquad 2(s_2^2 + \cdots + s_m^2) + s_{m+1}^2$$

subject to the constraints

$$2(s_2 + s_3 + \cdots + s_m) + s_{m+1} = 1, \quad 2s_2 \le s_{m+1}, \quad \text{and} \quad s_i \ge 0 \text{ for } 2 \le i \le m+1.$$

Extending $(s_2, \ldots, s_{m+1})$ by letting $s_{2m-i} = s_{2+i}$ for $0 \le i \le m-2$ relates this optimization problem to the symmetric probability vector $\mathbf{s}$ considered earlier. This problem incorporates the constraint (4), but it ignores the requirement in the original problem that $\mathbf{s}$ be realizable as the convolution of two probability vectors. It then suffices to show that we can realize the resulting optimum by such a convolution.

Such constrained optimization problems can be solved using the Karush-Kuhn-Tucker (KKT) conditions (see for example S. Boyd and L. Vandenberghe (2004), *Convex Optimization*, Cambridge University Press). Satisfying the conditions is sufficient for a global optimum. The method starts with a generalized Lagrangian incorporating the objective function, the inequality constraints, and the equality constraints:

$$L = 2(s_2^2 + \cdots + s_m^2) + s_{m+1}^2 + \mu(2s_2 - s_{m+1}) + \lambda\big(2(s_2 + \cdots + s_m) + s_{m+1} - 1\big).$$

The KKT conditions require partial derivatives with respect to the original variables and the multipliers for equality constraints to be 0, while for the multipliers of the inequality constraints we must have nonnegativity (see (9)) and "complementary slackness" (see (10)). That is,

$$\frac{\partial L}{\partial s_2} = 4s_2 + 2\mu + 2\lambda = 0; \tag{5}$$

$$\frac{\partial L}{\partial s_i} = 4s_i + 2\lambda = 0 \qquad \text{for } 3 \le i \le m; \tag{6}$$

$$\frac{\partial L}{\partial s_{m+1}} = 2s_{m+1} - \mu + \lambda = 0; \tag{7}$$

$$2(s_2 + \cdots + s_m) + s_{m+1} - 1 = 0. \tag{8}$$

$$\mu \ge 0; \quad \text{and} \tag{9}$$

$$\mu(2s_2 - s_{m+1}) = 0. \tag{10}$$

We also require $s_i \ge 0$ for all i in $\{2, \ldots, m+1\}$.

We show first that λ must be negative. If $\lambda > 0$, then by (6) each s_i with $i \ge 3$ is negative, which is forbidden. If $\lambda = 0$, then (6) requires $s_3 = \cdots = s_m = 0$. Since (5) now reads $4s_2 + 2\mu = 0$, it forbids $\mu > 0$, so $\mu = 0$ by (9). Now $s_2 = 0$ by (5) and $s_{m+1} = 0$ by (7), but that contradicts (8).

Hence $\lambda < 0$. Note that subtracting (5) from (7) gives $2s_{m+1} - 4s_2 = 3\mu + \lambda$. Since we require $2s_2 \le s_{m+1}$ and have $\lambda < 0$, we must have $\mu > 0$. Now (10) requires $2s_2 = s_{m+1}$.

With these restrictions, (5)–(7) reduce to

$$\lambda = -3\mu, \qquad s_2 = \mu, \qquad s_{m+1} = 2\mu, \qquad \text{and} \qquad s_i = \frac{3}{2}\mu \quad \text{for } 3 \le i \le m.$$

Using $s_{m+1} + 2\sum_{i=2}^{m} s_i = 1$, we obtain $\mu = 1/(3m-2)$, and consequently

$$s_2 = \frac{1}{3m-2}, \qquad s_{m+1} = \frac{2}{3m-2}, \qquad \text{and} \qquad s_i = \frac{3}{6m-4} \quad \text{for } 3 \le i \le m.$$

Extending back to the probability vector $\mathbf{s}$ with indices 2 through $2m$, we obtain

$$\mathbf{s} = \frac{1}{6m-4}(2, 3, 3, \ldots, 3, 3, 4, 3, 3, \ldots, 3, 3, 2), \tag{11}$$

yielding the minimum probability $\sum_{k=2}^{2m} s_k^2 = 3/(6m-4)$.

This solution to the optimization problem is achievable as the convolution of the two probability vectors

$$\left(\frac{1}{2}, 0, 0, \ldots, 0, 0, \frac{1}{2}\right) \qquad \text{and} \qquad \frac{1}{3m-2}(2, 3, 3, \ldots, 3, 3, 2).$$

Our final task is to show that these are the only probability vectors whose convolution is (11). To achieve $s_2 = s_{2m} > 0$, we have $p_1 = p_m > 0$ and $r_1 = r_m > 0$. Since we must satisfy

$$2s_2 = s_{m+1} = p_1 r_m + p_m r_1 + \sum_{i=2}^{m-1} p_i r_{m+1-i},$$

we obtain $p_i r_{m+1-i} = 0$ for $2 \le i \le m-1$. Consequently, for each i with $2 \le i \le m-1$,

$$p_i = p_{m+1-i} = 0 \quad \text{or} \quad r_{m+1-i} = r_i = 0.$$

By symmetry, we may take $p_2 = 0$. Now let k be the least integer in $\{2, \ldots, m\}$ such that $p_k > 0$. It suffices to show that $k = m$, which yields $\mathbf{p} = (1/2, 0, \ldots, 0, 1/2)$, whereupon the known convolution (11) yields $\mathbf{r}$ as claimed.

Suppose $k < m$. By (11),

$$\frac{3}{6m-4} = s_i = p_1 r_{i-1} + 0 + 0 + \cdots + 0 \qquad \text{for } 3 \le i \le k.$$

Since $p_1 r_1 = 2/(6m-4)$, we obtain $r_{i-1} = 3r_1/2 > 0$ for $3 \le i \le k$.

Next, $s_{k+1} = p_1 r_k + p_k r_1$. Since $p_k r_k = p_k r_{m+1-k} = 0$ and $p_k > 0$, we have $r_k = 0$. Now $p_k r_1 = s_{k+1} = 3/(6m-4)$ and $p_1 r_1 = s_2 = 2/(6m-4)$. Thus, $p_k = 3p_1/2$. Finally,

$$s_{k+2} \ge p_k r_2 = \left(\frac{3}{2}p_1\right)\left(\frac{3}{2}r_1\right) > 2s_2 = \frac{4}{6m-4},$$

contradicting $s_{k+2} \le 4/(6m-4)$. Thus $k = m$, completing the proof.

Editorial comment. The problem arose as an extension of Problem 1290 in Stan Wagon's Problem of the Week, which in turn was inspired by a problem on Tanya Khovanova's blog: blog.tanyakhovanova.com/2018/12/two-dice.

No solutions to part (b) or other correct solutions to part (a) were received.

A Lower Bound on Average Squared Acceleration

12229 [2021, 89]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $f : [0, 1] \to \mathbb{R}$ be a function that has a continuous second derivative and that satisfies $f(0) = f(1)$ and $\int_0^1 f(x)\,dx = 0$. Prove

$$30240 \left(\int_0^1 x f(x)\,dx\right)^2 \le \int_0^1 \left(f''(x)\right)^2\,dx.$$

Solution by Rory Molinari, Beverly Hills, MI. Applying integration by parts twice, and using $\int_0^1 f(x)\,dx = 0$ and $\int_0^1 f'(x)\,dx = f(1) - f(0) = 0$, we get

$$\int_0^1 x f(x)\,dx = \int_0^1 \left(x - \frac{1}{2}\right) f(x)\,dx = -\int_0^1 \left(\frac{x^2}{2} - \frac{x}{2}\right) f'(x)\,dx$$

$$= -\int_0^1 \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}\right) f'(x)\,dx = \int_0^1 \left(\frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}\right) f''(x)\,dx.$$

Thus, by the Cauchy–Schwarz inequality,

$$\left(\int_0^1 xf(x)\,dx\right)^2 = \left(\int_0^1 \left(\frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}\right) f''(x)\,dx\right)^2$$
$$\leq \left(\int_0^1 \left(\frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}\right)^2 dx\right) \cdot \left(\int_0^1 (f''(x))^2\,dx\right) = \frac{1}{30240}\int_0^1 (f''(x))^2\,dx,$$

and the desired conclusion follows.

Editorial comment. Justin Freeman generalized the problem by proving

$$\frac{(2n+2)!}{|B_{2n+2}|}\left(\int_0^1 xf(x)\,dx\right)^2 \leq \int_0^1 (f^{(n)}(x))^2\,dx,$$

where B_k is the kth Bernoulli number.

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), P. Bracken, B. Bradie, H. Chen, G. Fera (Italy), J. Freeman (Netherlands), K. Gatesman, G. Góral (Poland), N. Grivaux (France), L. Han, E. A. Herman, L. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), Z. Lin (China), J. H. Lindsey II, O. P. Lossers (Netherlands), I. Manzur (UK) & M. Graczyk (France), T. M. Mazzoli (Austria), A. Natian (UK), A. Pathak (India), B. Shala (Slovenia), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, J. Yan (China), L. Zhou, U. M. 6. P. MathClub (Morocco), and the proposer.

Families of Permutations with Equal Size

12230 [2021, 178]. *Proposed by David Callan, University of Wisconsin, Madison, WI.* Let $[n] = \{1, \dots, n\}$. Given a permutation $(\pi_1, \dots, \pi_n)$ of $[n]$, a *right-left minimum* occurs at position i if $\pi_j > \pi_i$ whenever $j > i$, and a *small ascent* occurs at position i if $\pi_{i+1} = \pi_i + 1$. Let $A_{n,k}$ denote the set of permutations π of $[n]$ with $\pi_1 = k$ that do not have right-left minima at consecutive positions, and let $B_{n,k}$ denote the set of permutations π of $[n]$ with $\pi_1 = k$ that have no small ascents.
(a) Prove $|A_{n,k}| = |B_{n,k}|$ for $1 \leq k \leq n$.
(b) Prove $|A_{n,j}| = |A_{n,k}|$ for $2 \leq j < k \leq n$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. For $n = 1$, we have $|A_{1,1}| = |B_{1,1}| = 1$. Hence it suffices to show that both $c_{n,k} = |A_{n,k}|$ and $c_{n,k} = |B_{n,k}|$ satisfy the recurrence

$$c_{n,k} = \begin{cases} \displaystyle\sum_{j=2}^{n-1} c_{n-1,j} & \text{if } k = 1, \\ \displaystyle\sum_{j=1}^{n-1} c_{n-1,j} & \text{if } k > 1. \end{cases}$$

The common recurrence then shows (**a**), and its form implies (**b**).

To a permutation π of $[n]$, associate the permutation σ of $[n-1]$ obtained by deleting π_1 and decreasing all entries exceeding π_1 by 1. From π_1 and σ, we can reconstruct π uniquely. In addition, σ has a right-left minimum at position i if and only if π has a right-left minimum at position $i + 1$.

For $k > 1$, any permutation σ of $[n-1]$ with no right-left minima in consecutive positions arises from a permutation $\pi \in A_{n,k}$, and permutations in $A_{n,k}$ generate such σ, since position 1 in π is not a right-left minimum. Thus, the recursive formula holds for $|A_{n,k}|$ when $k > 1$. When $k = 1$, π has a right-left minimum in position 1, so we must ensure that the corresponding σ has no right-left minimum in position 1, which is equivalent to $\sigma_1 \neq 1$. Thus, the formula holds also for $|A_{n,1}|$.

We show that this recurrence also holds for $B_{n,k}$. Again consider the same map, with $\pi \in B_{n,k}$. If σ has no small ascents, then also π has none, unless $\sigma_1 = k$. On the other hand, if π has no small ascents, then σ has at most one small ascent, with equality exactly when $\pi_j = k-1$ and $\pi_{j+1} = k+1$ for some j. Let $E_{n-1,k}$ be the set of permutations of $[n-1]$ with a small ascent involving entries $k-1$ and k and no other small ascents. We obtain

$$|B_{n,k}| = \begin{cases} \sum_{j=2}^{n-1} |B_{n-1,j}| & \text{if } k = 1, \\ |E_{n-1,k}| + \sum_{j \neq k} |B_{n-1,j}| & \text{if } 2 \leq k \leq n-1, \\ \sum_{j=1}^{n-1} |B_{n-1,j}| & \text{if } k = n. \end{cases}$$

We now prove $|E_{n-1,k}| = |B_{n-1,k}|$ when $n \geq 3$, which reduces this expression to the desired recurrence. Suppose $\sigma \in E_{n-1,k}$. Since σ has only one small ascent, the value $k+1$ does not follow k in σ. Hence collapsing the pair $(k-1, k)$ of consecutive values to $k-1$ and decreasing larger values by 1 gives a permutation of $[n-2]$ with no small ascent, and the map is reversible. Hence $|E_{n-1,k}| = \sum_{j=1}^{n-2} |B_{n-2,j}|$. We now have a proof of the desired recurrence by induction on n, since the induction hypothesis yields $|E_{n-1,k}| = |B_{n-1,k}|$.

Editorial comment. The proposer constructed a bijection from $A_{n,k}$ to $B_{n,k}$ iteratively as follows. If the current permutation has a small ascent, choose the left-most small ascent and move the larger value $j+1$ so that it immediately follows the largest right-left minimum m that it exceeds. For example, $\pi = (10, 11, 12, 2, 3, 1, 6, 7, 4, 8, 9, 5)$ has right-left minima at values 5, 4, and 1 (no two consecutive), and it has small ascents ending in the values 11, 12, 3, 7, and 9. The first iteration moves 11 to immediately after 5 and the fourth and final iteration yields (10, 12, 2, 1, 3, 6, 4, 8, 5, 7, 9, 11).

Yury Ionin observed that exchanging the values k and $k+1$ in $\pi \in A_{n,k}$ yields a bijection between $A_{n,k}$ and $A_{n,k+1}$ for $k > 1$. This is implicit in the featured solution.

Also solved by K. Gatesman, A. Goel, Y. J. Ionin, and the proposer. Part (**b**) also solved by N. Hodges (UK).

Complete Elliptic Integrals and Watson's Integrals

12232 [2021, 178]. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{x(1-x)}\sqrt{y(1-y)}\sqrt{1-xy}}\,dx\,dy = \frac{1}{4\pi}\left(\int_0^\infty e^{-t} t^{-3/4}\,dt\right)^4.$$

Solution I by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. Let I denote the integral on the left side of the desired equation. Substituting $x = k^2$ and $y = \sin^2 t$, we get

$$I = 4\int_0^1 \frac{1}{\sqrt{1-k^2}} \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 t}}\,dt\,dk = 4\int_0^1 \frac{K(k)\,dk}{\sqrt{1-k^2}}, \tag{1}$$

where $K(k)$ is the complete elliptic integral of the first kind given by the formula

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

The last integral in (1) is given by equation 6.143 on page 632 of I. S. Gradshteyn and I. M. Ryzhik (2007), *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press. Filling in its value, we obtain

$$I = 4\big(K(\sqrt{2}/2)\big)^2 = \frac{\big(\Gamma(1/4)\big)^4}{4\pi} = \frac{1}{4\pi}\left(\int_0^\infty e^{-t}t^{-3/4}\,dt\right)^4.$$

Solution II by Lixing Han, University of Michigan, Flint, MI, and Xinjia Tang, Changzhou University, Changzhou, China. Let I be as in Solution I. Substituting $x = \cos^2 u$, $y = \cos^2 v$, we get

$$I = 4\int_0^{\pi/2}\int_0^{\pi/2}\frac{du\,dv}{\sqrt{1-\cos^2 u\cos^2 v}} = \int_0^{\pi}\int_0^{\pi}\frac{du\,dv}{\sqrt{1-\cos^2 u\cos^2 v}}. \tag{2}$$

For $|a| < 1$, the substitution $s = \tan(t/2)$ yields

$$\int_0^\pi \frac{dt}{1-a\cos t} = \frac{2}{1-a}\int_0^\infty \frac{ds}{1+\frac{1+a}{1-a}s^2} = \frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\,s\right)\Bigg|_0^\infty = \frac{\pi}{\sqrt{1-a^2}}.$$

Setting $a = \cos u\cos v$ leads to

$$\int_0^\pi \frac{dt}{1-\cos u\cos v\cos t} = \frac{\pi}{\sqrt{1-\cos^2 u\cos^2 v}}.$$

Substituting into (2), we obtain

$$I = \frac{1}{\pi}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dt\,du\,dv}{1-\cos u\cos v\cos t} = \pi^2 I_1,$$

where I_1 is one of Watson's triple integrals (see I. J. Zucker (2011), 70+ years of the Watson Integrals, *J. Stat. Phys.* 145: 591–612, inp.nsk.su/~silagadz/Watson_Integral.pdf). Filling in the known value of I_1 gives the desired result.

Also solved by U. Abel & V. Kushnirevych (Germany), A. Berkane (Algeria), N. Bhandari (India), P. Bracken, H. Chen, B. E. Davis, G. Fera (Italy), M. L. Glasser, J.-P. Grivaux (France), J. A. Grzesik, N. Hodges (UK), Z. Lin (China), O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), M. Wildon (UK), and the proposer.

Squarefree Sums

12236 [2021, 179]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let p_k be the kth prime number, and let $a_n = \prod_{k=1}^n p_k$. Prove that for $n \in \mathbb{N}$ every positive integer less than a_n can be expressed as a sum of at most $2n$ distinct divisors of a_n.

Solution by Rory Molinari, Beverly Hills, MI. The divisors of a_n are exactly the positive squarefree integers whose largest prime factor is no bigger than p_n. We need the claim that every positive integer r can be written as the sum of at most two distinct positive squarefree integers.

It is easy to verify the claim for $r \le 9$, so assume $r \ge 10$. Let $A(r)$ be the set of positive squarefree integers not greater than r. If $r \in A(r)$, we are done. Otherwise, it is known that $|A(r)| \ge 53r/88$ for all r (see K. Rogers (1964), The Schnirelmann density of the squarefree integers, *Proc. Am. Math. Soc.* 15(4): 515–516). Thus $|A(r)| > 1 + r/2$ for $r \ge 10$, and the pigeonhole principle implies that $A(r)$ and $\{r - k : k \in A(r)\}$ share at least two elements. At least one of them is not $r/2$, yielding an expression of r as the sum of two elements of $A(r)$.

To prove the problem statement, we use induction on n. The claim holds trivially for $n = 1$. For $n > 1$, consider m such that $1 \le m < a_n$. Write m as $q \cdot p_n + r$ with $0 \le q < a_{n-1}$ and $0 \le r < p_n$. By the claim, r is the sum of at most two positive squarefree numbers. These numbers cannot have p_n as a factor since $r < p_n$, so they are factors of a_{n-1}. By the induction hypothesis, q is the sum of at most $2(n-1)$ distinct factors of a_{n-1}. Hence, $q \cdot p_n + r$ is the sum of at most $2(n-1)$ distinct divisors of a_n, all of which are multiples of p_n, plus at most two distinct divisors of a_{n-1}. It follows that m is the sum of at most $2n$ distinct divisors of a_n.

Editorial comment. The problem statement above corrects a typo in the original printing. All solvers used similar proofs. Some used bounds such as

$$|A(r)| \ge r - r \sum_{k=1}^{\infty} p_k^{-2} > .54r$$

in the proof of the initial claim.

Also solved by O. Geupel (Germany), N. Hodges (UK), M. Hulse (India), Y. J. Ionin, O. P. Lossers (Netherlands), C. Schacht, A. Stadler (Switzerland), M. Tang, R. Tauraso (Italy), and the proposer.

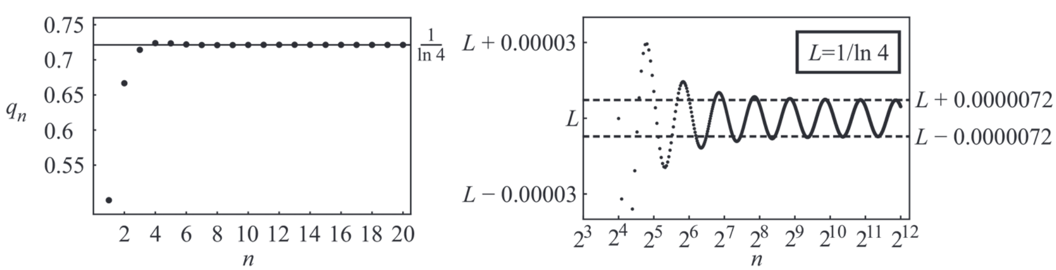
CLASSICS

C9. *From the 2001 Putnam Competition, suggested by the editors.* Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Flipping Coins Until They are All Heads

C8. *Due to Leonard Räde, suggested by the editors*. Start with n fair coins. Flip all of them. After this first flip, take all coins that show tails and flip them again. After the second flip, take all coins that still show tails and flip them again. Repeat until all coins show heads. Let q_n be the probability that the last flip involved only a single coin. What is $\lim_{n\to\infty} q_n$?

Solution. Let $L = 1/\ln 4$. Rough computation suggests that q_n converges to L, but we show that q_n oscillates around L with an asymptotic amplitude of about 10^{-5}, and so the limit does not exist. Here at left we display the graph of q_n for $1 \le n \le 20$, illustrating the apparent convergence. At right we graph the same sequence, zooming in and using a logarithmic horizontal axis. That view reveals what appears to be a persistent asymptotic oscillation.



To prove that the limit does not exist, take $n \ge 2$, let C be one of the coins, and let k be a positive integer. Consider the event that C shows heads for the first time on flip $k + 1$, and all other coins show heads earlier. This occurs only if C shows tails for each of the first k flips and then heads on flip $k + 1$. This has probability $2^{-(k+1)}$. For each of the other $n - 1$ coins, it must not be the case that all of the first k flips show tails. This has probability $1 - 2^{-k}$. So the probability of the event is $2^{-(k+1)}(1 - 2^{-k})^{n-1}$.

Because there are n possibilities for C, and because k can be any positive integer,

$$q_n = \sum_{k=1}^{\infty} \frac{n}{2^{k+1}} \left(1 - \frac{1}{2^k}\right)^{n-1}. \tag{$*$}$$

We show that the sequence $q_1, q_2, \dots$ does not converge by showing that it has different subsequences that converge but to different limits.

Let $c_k = (1 - 2^{-k})^{2^k}$. It is well known and easy to show that $c_1, c_2, \dots$ is an increasing sequence and $\lim_{k\to\infty} c_k = 1/e$.

We have

$$q_n = \sum_{k=1}^{\infty} \frac{n}{2^{k+1}} \left(\left(1 - \frac{1}{2^k}\right)^{2^k}\right)^{n/2^k} \left(1 - \frac{1}{2^k}\right)^{-1} = \sum_{k=1}^{\infty} \frac{n}{2^{k+1}}\, c_k^{n/2^k} \left(\frac{2^k}{2^k - 1}\right).$$

Now fix an odd integer m, and let $a_j = q_{m2^j}$ for $j \ge 1$. We have

$$a_j = \sum_{k=1}^{\infty} \frac{m2^j}{2^{k+1}}\, c_k^{m2^j/2^k} \left(\frac{2^k}{2^k - 1}\right) = \sum_{k=1-j}^{\infty} \frac{m}{2^{k+1}}\, c_{k+j}^{m/2^k} \left(\frac{2^{k+j}}{2^{k+j} - 1}\right).$$

The kth term of this series is bounded above by $(m/2^k)e^{-m/2^k}$, whose sum over k from $-\infty$ to ∞ is finite. Hence, by the dominated convergence theorem,

$$\lim_{j\to\infty} a_j = \sum_{k=-\infty}^{\infty} \lim_{j\to\infty} \frac{m}{2^{k+1}}\, c_{k+j}^{m/2^k} \left(\frac{2^{k+j}}{2^{k+j} - 1}\right) = \sum_{k=-\infty}^{\infty} \frac{m}{2^{k+1}} e^{-m/2^k}.$$

With $m = 1$, this last sum can be approximated by letting k run from -5 to 27, giving an approximation of $L + 4.58 \cdot 10^{-6}$ for the sum, and the error in this approximation is seen by a simple integration to be less than 10^{-8}. Similarly, when $m = 3$, the last sum is approximately $L - 1.17 \cdot 10^{-6}$, again with an error of less than 10^{-8}. The distinct limits prove that $\lim_{n\to\infty} q_n$ does not exist.

Editorial comment. One can approximate the sum in $(*)$ by

$$\int_0^{\infty} n2^{-(x+1)}(1 - 2^{-x})^{n-1}\, dx,$$

which is L, independent of n. The error in this approximation does not vanish with n, however.

The problem appeared in this MONTHLY [1991, 366; 1994, 78]. A version of the same problem appeared almost a decade earlier in the 1982 *Can. Math. Bull.* as Problem P322 by George Szekeres, who asked whether

$$\lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{i-1} \frac{i}{2^i - 1} \binom{n}{i}$$

equals $1/\ln 2$. It turns out that the nth term here is just $2q_n$ in disguise, so the answer to the Szekeres problem is negative.

In N. J. Calkin, E. R. Canfield, and H. S. Wilf (2000), Averaging sequences, deranged mappings, and a problem of Lambert and Slater, *J. Comb. Th., Ser. A* 91(1–2): 171–190, a general class of sequences is found to exhibit the oscillating sequence phenomenon. In particular, they answer an open question in D. E. Lampert and P. J. Slater (1998), Parallel knockouts in the complete graph, this MONTHLY 105: 556–558.