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Daniel H. Ullman Edited by, Daniel J. Velleman, Stan Wagon, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

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Proposed problems, solutions, and classics should be submitted online at americanmathematicalmonthly.submittable.com/submit. *Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by April 30, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

12356. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* Let $A(z) = z^3 - z^2$ and $B(z) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\binom{3n+1}{n} z^{n+1}$. Prove that B is a one-sided inverse to A in the sense that $A(B(z)) = z$. Also, prove $B(A(z)) = 1 - z^2 M(-z)$, where

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

(The coefficients of $M(z)$ are the Motzkin numbers 1, 1, 2, 4, 9, 21, ...)

12357. *Proposed by Van Khea, Prey Veng, Cambodia, and Dan Ştefan Marinescu, Hunedoara, Romania.* Suppose that triangles ABC and DEF have the same centroid, where D, E, and F are on the segments BC, CA, and AB, respectively. Let I be the incenter of triangle ABC. Prove

$$\frac{AI}{AD} + \frac{BI}{BE} + \frac{CI}{CF} \le 2.$$

12358. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* For a positive integer q and a set A of positive integers, say that A is q-good if every sufficiently large integer has exactly q representations as the sum of distinct elements of A.

(a) Which sets A are 1-good?

(b) For which q does there exist a q-good set?

(c) For q as in **(b)**, which sets A are q-good?

12359. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Let n be a positive integer. Prove

$$\frac{-1-\pi}{4n} - \frac{1}{8n^2} < \sum_{k=1}^{n} \frac{1}{(2k-1)^2} - 2\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}\right)^2 < \frac{-1+\pi}{4n} - \frac{1}{8n^2}.$$

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12360. *Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania.* Evaluate

$$\lim_{n\to\infty} \frac{(n+1)^2}{x_{n+1}} - \frac{n^2}{x_n},$$

where $x_n = \sqrt[n]{\sqrt{2!}\sqrt[3]{3!}\cdots\sqrt[n]{n!}}$.

12361. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For a nonengative integer k, let $r_{3k} = 0$, $r_{3k+1} = 1$, and $r_{3k+2} = -1$. Prove

$$\sum_{k=0}^{n-1} \binom{2k}{k} = \sum_{k=0}^{n} r_k \binom{2n}{n-k},$$

for every positive integer n.

12362. *Proposed by Antonio Garcia, Strasbourg, France.* Evaluate

$$\lim_{n\to\infty} \int_0^{\pi/2} \frac{n}{\left(\sqrt{2}\cos x\right)^n + \left(\sqrt{2}\sin x\right)^n}\, dx.$$

SOLUTIONS

Counting Sets Without Consecutive Elements

12233 [2021, 178]. *Proposed by C. R. Pranesachar, Indian Institute of Science, Bengaluru, India.* Let n and k be positive integers with $1 \le k \le (n+1)/2$. For $1 \le r \le n$, let $h(r)$ be the number of k-element subsets of $\{1, \dots, n\}$ that do not contain consecutive elements but that do contain r. For example, with $n = 7$ and $k = 3$, the string $h(1), \dots, h(7)$ is 6, 3, 4, 4, 4, 3, 6. Prove
(a) $h(r) = h(r+1)$ when $r \in \{k, \dots, n-k\}$.
(b) $h(k-1) = h(k) \pm 1$.
(c) $h(r) > h(r+2)$ when $r \in \{1, \dots, k-2\}$ and r is odd.
(d) $h(r) < h(r+2)$ when $r \in \{1, \dots, k-2\}$ and r is even.

Composite solution by Kyle Gatesman, Johns Hopkins University, Baltimore, MD, and Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy. The problem statement requires correction in parts **(c)** and **(d)**, where in the special case $k = (n+1)/2$ we have $h(r) = h(r+2)$ for all r.

For a proof by induction, we make the dependence on n and k explicit. Let $h_{n,k}(r) = h(r)$, and extend the definition to give 0 when n, k, or r is outside its natural domain. For $1 \le r \le n-1$, partition the k-element subsets containing r by whether they contain n, obtaining

$$h_{n,k}(r) = h_{n-1,k}(r) + h_{n-2,k-1}(r). \tag{1}$$

Similarly, for $1 < r \le n$, partition the k-element subsets containing r by whether they contain 1. After shifting indices to start at 2 or 3, this yields

$$h_{n,k}(r) = h_{n-1,k}(r-1) + h_{n-2,k-1}(r-2). \tag{2}$$

(a) We use induction on n. Note that $h_{n,1}(r) = 1$ for all r and n, from which **(a)** follows for $k = 1$, including all cases with $n \le 3$. Now suppose $n > 3$ and $k > 1$. By symmetry,

$h_{n,k}(r) = h_{n,k}(n+1-r)$, so we need only consider $k \le r \le (n-1)/2$. In that case, $r \le (n-1) - k = (n-2) - (k-1)$. Now (1) and the induction hypothesis imply

$$h_{n,k}(r) = h_{n-1,k}(r) + h_{n-2,k-1}(r) = h_{n-1,k}(r+1) + h_{n-2,k-1}(r+1) = h_{n,k}(r+1).$$

(b) We use induction on k to prove that $h_{n,k}(k-1) - h_{n,k}(k) = (-1)^k$, for all positive integers n beginning with $h_{n,1}(0) = 0$ and $h_{n,1}(1) = 1$. By (1) and (2),

$$\begin{aligned} h_{n,k}(r) - h_{n,k}(r+1) &= \big(h_{n-1,k}(r) + h_{n-2,k-1}(r)\big) - \big(h_{n-1,k}(r) + h_{n-2,k-1}(r-1)\big) \\ &= -\big(h_{n-2,k-1}(r-1) - h_{n-2,k-1}(r)\big). \end{aligned} \tag{3}$$

With $r = k-1 \le ((n-2)+1)/2$, the induction hypothesis completes the proof.

(c, d) We use induction on r. The number of k-element subsets of $\{1, \dots, n\}$ having no consecutive elements is $\binom{n-k+1}{k}$, corresponding to insertions of k balls in distinct positions between or outside $n-k$ markers in a row. Thus $h_{n,k}(1) = \binom{n-k}{k-1}$, $h_{n,k}(2) = \binom{n-k-1}{k-1}$, and, by (2), $h_{n,k}(3) = \binom{n-k-2}{k-1} + \binom{n-k-1}{k-2}$. Using Pascal's formula for binomial coefficients twice, $h_{n,k}(1) - h_{n,k}(3) = \binom{n-k-2}{k-2}$. Thus $h_{n,k}(1) - h_{n,k}(3) > 0$ unless $k = (n+1)/2$, in which case the difference is 0. This completes the proof for $r = 1$.

Now suppose $r \ge 2$. If $k = (n+1)/2$, then n is odd, and $h_{n,k}(r)$ is 1 when r is odd and 0 when r is even, so the desired difference is 0. Hence we may restrict our attention to $k \le n/2$, which yields $k-1 \le (n-3+1)/2$. Using (1) and (2), then (3), and finally (1) and (2) again, we find

$$\begin{aligned} h_{n,k}(r) - h_{n,k}(r+2) &= h_{n-1,k}(r) + h_{n-2,k-1}(r) - h_{n-1,k}(r+1) - h_{n-2,k-1}(r) \\ &= -\big(h_{n-3,k-1}(r-1) - h_{n-3,k-1}(r)\big) \\ &= -\big(h_{n-2,k-1}(r-1) - h_{n-2,k-1}(r+1)\big). \end{aligned}$$

Now the induction hypothesis completes the proof.

Editorial comment. Nigel Hodges conditioned on the number j of selected elements preceding r to prove

$$h(r) = \sum_{j=0}^{k-1} \binom{r-1-j}{j} \binom{n-r-k+1+j}{k-1-j}.$$

He then used induction and Pascal's formula to prove for $r \le n-k+1$ that this expression equals $\sum_{j=0}^{r-1} (-1)^j \binom{n-k-j}{k-1-j}$, from which **(a)–(d)** all follow quickly.

Also solved by H. Chen (China), C. Curtis & J. Boswell, N. Hodges (UK), Y. J. Ionin, O. P. Lossers (Netherlands), L. J. Peterson, R. Stong, and the proposer.

A Congruence for a Product of Quadratic Forms

12234 [2021, 179]. *Proposed by Nicolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Let p be an odd prime, and let $Ax^2 + Bxy + Cy^2$ be a quadratic form with A, B, and C in $\mathbb{Z}$ such that $B^2 - 4AC$ is neither a multiple of p nor a perfect square modulo p. Prove that

$$\prod_{0<x<y<p} (Ax^2 + Bxy + Cy^2)$$

is 1 modulo p if exactly one or all three of A, C, and $A+B+C$ are perfect squares modulo p and is -1 modulo p otherwise.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. All expressions below involving x and y take place in the finite field $\mathbb{F}_p$ with p elements. We first study the desired product in general, leaving until later a consideration of how many elements of $\{A, C, A + B + C\}$ are squares. For convenience, define

$$Q(x, y) = Ax^2 + Bxy + Cy^2.$$

Since we are given that $B^2 - 4AC$ is a nonsquare, A and C must be nonzero, and it follows that $Q(x, y) \neq 0$ when $(x, y) \neq (0, 0)$. In order to evaluate the product $\prod_{0<x<y<p} Q(x, y)$, we want to group the factors by the value of $Q(x, y)$. That is, for each D we seek the number of solutions of $Q(x, y) = D$ such that $0 < x < y < p$.

For $D \neq 0$, since $Q(x, y) - Dz^2 = 0$ determines a nondegenerate quadric, there are altogether $p^2 - 1$ solution triples (x, y, z) to $Q(x, y) - Dz^2 = 0$. (See Lemma 7.23 on p. 142 of J. W. P. Hirschfeld (1979), *Projective Geometries over Finite Fields*, Clarendon Press.) The set of solution triples is invariant under multiplication by any nonzero element of $\mathbb{F}_p$. Hence the solutions come in $p + 1$ multiplicative classes of size $p - 1$, each containing one triple of the form $(x, y, 1)$, yielding $p + 1$ solutions to $Q(x, y) = D$.

This partitions the set of nonzero pairs (x, y) by the value of $Q(x, y)$, with each value D occurring exactly $p + 1$ times. Note that $Q(x, y) = Q(p - x, p - y)$, so for fixed D the number of pairs satisfying $Q(x, y) = D$ with $x < y$ equals the number of pairs with $x > y$. Hence we will need to divide the number of occurrences of D by 2.

Since we require $0 < x < y < p$ in the stated product, we must also exclude occurrences of D that arise when $x = 0$, $y = 0$, or $x = y$. Two nonzero elements of $\mathbb{F}_p$ have the same *quadratic character* if they are both squares or both nonsquares, equivalent to their ratio being a square. Occurrences of D on the line $x = 0$ have $Cy^2 - D = 0$, or $y^2 = D/C$, so there are two such pairs yielding D when D and C have the same quadratic character; otherwise none. Similarly, there are two occurrences of D on $y = 0$ if and only if A and D have the same quadratic character (satisfying $x^2 = D/A$), and two occurrences of D on $x = y$ if and only if $A + B + C$ and D have the same quadratic character (satisfying $x^2 = D/(A + B + C)$). Also, such occurrences on the three lines are distinct.

Let the number of squares among $\{A, C, A + B + C\}$ be s. Starting with the $p + 1$ pairs $(x, y) \in \mathbb{F}_p^2 - (0, 0)$ that generate D, we subtract the occurrences with $x = 0$, $y = 0$, or $x = y$ and then divide the remaining occurrences by 2, as discussed above. We thus compute that each square D occurs in the product $(p + 1 - 2s)/2$ times, while each nonsquare D occurs in the product $\big(p + 1 - 2(3 - s)\big)/2$ times.

This tells us how many times we have the product of all the squares and how many times we have the product of all the nonsquares. It is well known that the product of all the squares is $(-1)^{(p+1)/2}$, and the product of all the nonsquares is $(-1)^{(p-1)/2}$, because an element and its reciprocal have the same quadratic character. After canceling reciprocal pairs and ignoring 1, we are left with -1, which is a square if and only if $p \equiv 1 \bmod 4$.

We thus compute

$$\begin{aligned}
\prod_{0<x<y<p} Q(x, y) &= (-1)^{\frac{1}{2}(p+1)\frac{1}{2}(p+1-2s)}(-1)^{\frac{1}{2}(p-1)\frac{1}{2}(p+1+2s-6)} \\
&= (-1)^{\frac{1}{4}\left((p+1)^2+(p^2-1)-4s-6(p-1)\right)} \\
&= (-1)^{\frac{1}{2}(p^2-2p+3-2s)} = (-1)^{\frac{1}{2}\left((p-1)^2+2-2s\right)} = (-1)^{1-s}.
\end{aligned}$$

This equals 1 or -1 when the number s of squares in $\{A, C, A + B + C\}$ is odd or even, respectively, as desired.

Also solved by C. Curtis & J. Boswell, Y. J. Ionin, R. Tauraso (Italy), and the proposer.

An Application of Liouville's Theorem

12235 [2021, 179]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $a_0, a_1, \ldots$ be a sequence of real numbers tending to infinity, and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function satisfying

$$|f^{(n)}(a_k)| \le e^{-a_k}$$

for all nonnegative integers k and n. Prove $f(z) = ce^{-z}$ for some constant $c \in \mathbb{C}$ with $|c| \le 1$.

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. We prove that the entire function $g(z) = e^z f(z)$ satisfies

$$|g(z)| \le 1 \qquad (*)$$

for all z. From this, Liouville's theorem yields $g(z) = c$ for some constant c, and then $(*)$ yields $|c| \le 1$. Hence, $f(z) = ce^{-z}$ with $|c| \le 1$, as claimed.

Since $f(z)$ is entire, for $z = x + iy$ and $k \ge 0$ we have

$$\begin{aligned}|g(z)| = |e^z| \left| \sum_{n=0}^{\infty} \frac{f^{(n)}(a_k)}{n!}(z - a_k)^n \right| &\le e^x \sum_{n=0}^{\infty} \frac{|f^{(n)}(a_k)|}{n!}|z - a_k|^n \\ &\le e^x e^{-a_k} \sum_{n=0}^{\infty} \frac{|z - a_k|^n}{n!} = e^{x - a_k + |z - a_k|}.\end{aligned}$$

Since $\lim_{k\to\infty} a_k = \infty$, we have $x < a_k$ for sufficiently large k. Thus, for such k,

$$|g(z)| \le e^{|z - a_k| - |x - a_k|} = \exp\left(\frac{y^2}{|z - a_k| + |x - a_k|}\right).$$

Taking the limit as $k \to \infty$, we obtain $(*)$, which completes the proof.

Also solved by P. Bracken, L. Han (USA) & X. Tang (China), E. A. Herman, K. T. L. Koo (China), O. Kouba (Syria), K. Sarma (India), A. Sasane (UK), A. Stadler (Switzerland), J. Yan (China), and the proposer.

The Googolth Term of a Sequence

12237 [2021, 276]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let $x_0 = 1$ and $x_{n+1} = x_n + \lfloor x_n^{3/10} \rfloor$ for $n \ge 0$. What are the first 40 decimal digits of x_n when $n = 10^{100}$?

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The first 40 digits are 43236 87954 44259 51263 21573 91617 78825 77073.

Let $f(x) = (10/7)x^{7/10}$, and let $a_k = f(x_k)$ for all k. Applying the mean value theorem to f yields $c_n \in (x_n, x_{n+1})$ such that

$$a_{n+1} - a_n = c_n^{-3/10}(x_{n+1} - x_n) = c_n^{-3/10}\lfloor x_n^{3/10} \rfloor.$$

Since $c_n > x_n$, this implies $a_{n+1} - a_n < 1$. Computing $x_6 = 7$ and $a_6 = 10 \cdot 7^{-3/10} < 6$, we obtain $a_n < n$ and hence $x_n < (7n/10)^{10/7}$ for $n \ge 6$. Putting $n = 10^{100}$, we obtain an upper bound for x_n less than

$$4.3236\ 87954\ 44259\ 51263\ 21573\ 91617\ 78825\ 77073\ 38123 \times 10^{142}.$$

We now provide a lower bound for x_n. Applying the mean value theorem to $g(x) = x^{3/10}$ yields $b_n \in (x_n, x_{n+1})$ such that

$$c_n^{3/10} - x_n^{3/10} < x_{n+1}^{3/10} - x_n^{3/10} = \frac{3}{10} b_n^{-7/10}(x_{n+1} - x_n) = \frac{3}{10} b_n^{-7/10}\lfloor x_n^{3/10} \rfloor < 1.$$

Hence

$$a_{n+1} - a_n = 1 - \frac{c_n^{3/10} - \lfloor x_n^{3/10} \rfloor}{c_n^{3/10}} > 1 - \frac{2}{x_n^{3/10}}. \tag{$*$}$$

By direct iteration, $x_{45} = 102 > 4^{10/3}$. Since $\langle x_n \rangle$ is increasing, $a_{n+1} \geq a_n + 1/2$ whenever $n \geq 45$. From $a_{45} > 45/2$, for $n \geq 45$ we conclude that $a_n > n/2$, hence $x_n > (7n/20)^{10/7}$. Explicit computation shows that this lower bound for x_n also holds for $n < 45$. Therefore, summing ($*$) from 1 through $n - 1$ gives

$$a_n > a_1 + (n-1) - \sum_{k=1}^{n-1} \frac{2}{x_k^{3/10}} > n - \sum_{k=1}^{n-1} \frac{2}{(7k/20)^{3/7}} > n - \frac{7}{2(7/20)^{3/7}}\, n^{4/7},$$

where at the last step we used the standard integral bound

$$\sum_{k=1}^{n-1} \frac{1}{k^{3/7}} \leq \int_0^n \frac{1}{t^{3/7}}\, dt = \frac{7}{4}\, n^{4/7}.$$

For $n = 10^{100}$, this yields a lower bound for x_n greater than

$$4.3236\ 87954\ 44259\ 51263\ 21573\ 91617\ 78825\ 77073\ 37651 \times 10^{142}.$$

Therefore, the first 40 digits of x_n when $n = 10^{100}$ are as claimed.

Also solved by O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Tauraso (Italy), E. Treviño, T. Wilde (UK), The Logic Coffee Circle (Switzerland), and the proposer.

Collinear Midpoints from a Glide Reflection

12238 [2021, 276]. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a convex quadrilateral with $AD = BC$. Let P be the intersection of the diagonals AC and BD, and let K and L be the circumcenters of triangles PAD and PBC, respectively. Show that the midpoints of segments AB, CD, and KL are collinear.

Solution by Michel Bataille, Rouen, France. Let E and F be the midpoints of AB and CD, respectively. Let m be the line through D that is parallel to EF, and let m' be the image of m under reflection through EF. Since F is the midpoint of CD, the point C must lie on m'. Let Γ be the circle centered at B with radius AD. Since $AD = BC$, the point C also lies on Γ.

Consider the 180° rotation of the plane centered at E. This rotation sends A to B and D to some point D'. The rotation sends m to m', so D' lies on m', and since $BD' = AD$, the point D' also lies on Γ. However, D' cannot be C, because the midpoint of $D'D$ is E, whereas the midpoint of CD is F. Thus Γ and m' intersect at two points, and those two points are C and D'. It follows that if n is the line through B that is perpendicular to EF, then C is the reflection of D' through n.

Let g be the transformation of the plane consisting of rotation by 180° centered at E followed by reflection through n. One sees easily that g is an orientation-reversing isometry that sends A to B and D to C. (The transformation g can also be described as a glide reflection with axis EF.)

For any lines ℓ and ℓ', let $\angle(\ell, \ell')$ denote the directed angle from ℓ to ℓ'. Let Γ_{AD} and Γ_{BC} be the circumcircles of $\triangle PAD$ and $\triangle PBC$, respectively, and let $Q = g(P)$.

Since g is orientation-reversing, $\angle(QB, QC) = \angle(PD, PA) = \angle(PB, PC)$. Therefore Q lies on Γ_{BC}. However, also Q, B, and C lie on $g(\Gamma_{AD})$, so $g(\Gamma_{AD}) = \Gamma_{BC}$. It follows that $g(K) = L$, and therefore the midpoint of KL lies on EF.

Editorial comment. This solution shows that the quadrilateral need not be convex. Indeed, it need not even be simple, as long as the lines AC and BD intersect.

Also solved by A. Ali (India), J. Cade, H. Chen (China), P. De (India), G. Fera (Italy), D. Fleischman, K. Gatesman, O. Geupel (Germany), J.-P. Grivaux (France), W. Janous (Austria), D. Jones & M. Getz, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), T. Wiandt, L. Wimmer (Germany), L. Zhou, Davis Problem Solving Group, and the proposer.

Factorials and Powers of 2

12239 [2021, 276]. *Proposed by David Altizio, University of Illinois, Urbana, IL.* Determine all positive integers r such that there exist at least two pairs of positive integers (m, n) satisfying the equation $2^m = n! + r$.

Solution by Celia Schacht, North Carolina State University, Raleigh, NC. There are two such values of r. They are $r = 2$, with $2^3 = 3! + 2$ and $2^2 = 2! + 2$, and $r = 8$, with $2^7 = 5! + 8$ and $2^5 = 4! + 8$. We show that there are no other values.

If $2^{m_1} = n_1! + r$ and $2^{m_2} = n_2! + r$, then $2^{m_1} - n_1! = 2^{m_2} - n_2!$. For $x \in \mathbb{N}$, let $2^{v(x)}$ be the highest power of 2 dividing x. Note that x can be uniquely written as $2^{v(x)}$ times an odd number, which we call the odd part of x. Since $r > 0$, we have $2^{m_i} > n_i!$, so $m_i > v(n_i!)$ for $i \in \{1, 2\}$. Therefore,

$$v(n_1!) = v(2^{m_1} - n_1!) = v(2^{m_2} - n_2!) = v(n_2!).$$

Given that $(m_1, n_1) \neq (m_2, n_2)$, we may assume $m_1 > m_2$ and $n_1 > n_2$. If there are any even numbers from $n_2 + 1$ to n_1, then $v(n_1!) > v(n_2!)$, so $v(n_1!) = v(n_2!)$ implies that n_2 is even and $n_1 = n_2 + 1$. Let $n_2 = 2k$. Thus

$$2^{m_1} - 2^{m_2} = n_1! - n_2! = (2k) \cdot (2k)!. \tag{4}$$

The odd part of the left side is $2^{m_1 - m_2} - 1$. It equals the product of the odd parts of $2k$ and $(2k)!$, so it is at least the odd part of $(2k)!$, which we write as $2q + 1$. That is, $2^{m_1 - m_2} - 1 \geq 2q + 1$.

By dividing out all the factors of 2 from $(2k)!$, we obtain

$$v((2k)!) = \sum_{i=1}^{\infty} \left\lfloor \frac{2k}{2^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{2k}{2^i} = 2k.$$

First consider the case $k \geq 5$. By induction, $(2k)! > 2^{4k}$ for $k \geq 5$. Therefore,

$$2^{4k} < (2k)! = 2^{v((2k)!)}(2q + 1) < 2^{2k}(2q + 1),$$

so $2^{2k} - 1 < 2^{2k} < 2q + 1 \leq 2^{m_1 - m_2} - 1$. Also $(2k)! = n_2! < n_2! + r = 2^{m_2}$, which yields

$$(2k)!\left(2^{2k} - 1\right) < 2^{m_2}\left(2^{2k} - 1\right) < 2^{m_1} - 2^{m_2} = (2k) \cdot (2k)!.$$

Dividing by $(2k)!$ yields $2^{2k} - 1 < 2k$, which is false for all positive k. This contradiction eliminates the possibility $k \geq 5$.

It remains to check the cases of the form $(n_1, n_2) = (2k + 1, 2k)$ for $k \in \{1, 2, 3, 4\}$. According to (4), we need powers of 2 differing by $2k(2k)!$. For $1 \leq k \leq 4$, the values of $2k(2k)!$ are 4, 96, 4320, and 322560, respectively. Examining powers of 2 yields the solutions for $k \in \{1, 2\}$ listed at the start, but no solution for $k \in \{3, 4\}$.

Also solved by A. Ali (India), F. R. Ataev (Uzbekistan), C. Curtis & J. Boswell, S. M. Gagola Jr., K. Gatesman, M. Ghelichkhani (Iran), N. Hodges (UK), P. Komjáth (Hungary), O. P. Lossers (Netherlands), S. Omar (Morocco), J. Polo-Gómez (Canada), K. Sarma (India), A. Stadler (Switzerland), R. Stong, M. Tang, R. Tauraso (Italy), E. Treviño, T. Wilde (UK), L. Zhou, and the proposer.

Harmonic Sums: Euler Once, Abel Twice

12241 [2021, 276]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Prove

$$\sum_{n=1}^{\infty}(-1)^n\, n\left(\frac{1}{4n}-\ln 2+\sum_{k=n+1}^{2n}\frac{1}{k}\right)=\frac{\ln 2-1}{8}.$$

Solution by Kee-Wai Lau, Hong Kong, China. We first address the partial sum of the series on the left side and show

$$\begin{aligned}8\sum_{n=1}^{N}(-1)^n n\left(\frac{1}{4n}-\ln 2+\sum_{k=n+1}^{2n}\frac{1}{k}\right) \qquad\qquad (1)\\ =2(-1)^N(2N+1)\left(\sum_{k=N+1}^{2N}\frac{1}{k}-\ln 2\right)+\sum_{n=1}^{N}\frac{(-1)^n}{n}+(-1)^N-1+2\ln 2.\end{aligned}$$

Since $\ln 2$ is irrational, it must have the same coefficient on both sides, requiring

$$8\sum_{n=1}^{N}(-1)^n n=2(-1)^N(2N+1)-2.$$

This equality is easily verified by considering odd and even N separately. Let $K(N)$ denote the quantity on both sides. In addition, since $8\sum_{n=1}^{N}(-1)^n(1/4)=(-1)^N-1$, the sum of the N initial terms on the left in (1) equals the sum of two terms on the right. It remains to prove

$$\sum_{n=1}^{N}8(-1)^n n\sum_{k=n+1}^{2n}\frac{1}{k}=2(-1)^N(2N+1)\sum_{k=N+1}^{2N}\frac{1}{k}+\sum_{n=1}^{N}\frac{(-1)^n}{n}.$$

Let $L(N)$ denote the left side in this equation. Rewrite that double sum as

$$L(N)=\sum_{n=1}^{N}(K(n)-K(n-1))J(n),$$

where $J(n)=\sum_{k=n+1}^{2n}1/k$ and $K(0)=0$. By partial summation,

$$L(N)=K(N)J(N)+\sum_{n=1}^{N-1}K(n)(J(n)-J(n+1)).$$

Now

$$J(n)-J(n+1)=\frac{1}{n+1}-\frac{1}{2n+1}-\frac{1}{2n+2}=\frac{-1}{2(n+1)(2n+1)}.$$

Hence

$$\begin{aligned}L(N)&=\big(2(-1)^N(2N+1)-2\big)J(N)+\sum_{n=1}^{N-1}\big((-1)^{n+1}(2n+1)+1\big)\frac{1}{(n+1)(2n+1)}\\ &=2(-1)^N(2N+1)J(N)+\sum_{n=1}^{N-1}\frac{(-1)^{n+1}}{n+1}-2J(N)+\sum_{n=1}^{N-1}\frac{1}{(n+1)(2n+1)}. \qquad (2)\end{aligned}$$

Restoring the expression involving J in the last summand, the last two terms in (2) simplify by telescoping as

$$-2J(N)-2\sum_{n=1}^{N-1}(J(n)-J(n+1))=-2J(N)-2(J(1)-J(N))=-1.$$

Now the expression for $L(N)$ reduces to the right side of (1), completing the proof of the identity.

Let H_N denote the harmonic number $\sum_{n=1}^{N} 1/n$. By Euler–Maclaurin summation,

$$H_N = \ln N + \gamma + \frac{1}{2N} + O(N^{-2}),$$

where γ is Euler's constant. Thus

$$\sum_{n=N+1}^{2N} \frac{1}{n} = H_{2N} - H_N = \ln 2 - \frac{1}{4N} + O(N^{-2}).$$

Hence the first term on the right side of (1) simplifies as

$$2(-1)^N(2N+1)\left(\frac{-1}{4N} + O(N^{-2})\right) = -(-1)^N + O(N^{-1}).$$

Also,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2.$$

Thus the right side of $(*)$ converges to $-1 + \ln 2$, which completes the proof.

Editorial comment. Another approach to evaluating the left side is to introduce the factor x^n for $0 < x < 1$ into the sum, expand, and let x approach 1. This is an application of Abel's limit theorem, known as Abel summation. Ulrich Abel (fittingly) and Vitaliy Kushnirevych used this method. With

$$a_n = \frac{1}{4n} - \ln 2 + H_{2n} - H_n \qquad \text{and} \qquad g(x) = \sum_{n=1}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x},$$

let

$$f(x) = \sum_{n=1}^{\infty} a_n(-x)^n = \frac{-\ln(1+x)}{4} - \frac{x\ln 2}{1+x} + \frac{g(i\sqrt{x}) + g(-i\sqrt{x})}{2} - g(-x).$$

Upon differentiating $f(x)$, we obtain a power series for $(-1)^n n a_n$, and Abel summation yields the result.

Many solvers used a method somewhat akin to Abel summation, that of integral representation. For example, Richard Stong used

$$a_n = \frac{1}{2}\int_0^1 \frac{1-x}{1+x} x^{2n-1}\, dx.$$

Upon interchange of summation and integration (justified by dominated convergence), the desired sum then becomes the readily evaluated integral

$$-\frac{1}{2}\int_0^1 \frac{1-x}{1+x}\frac{x}{(1+x^2)^2}\, dx.$$

Also solved by U. Abel & V. Kushnirevych (Germany), A. Berkane (Algeria), P. Bracken, B. Bradie, H. Chen, G. Fera (Italy), K. Gatesman, M. L. Glasser, G. C. Greubel, L. Han (US) & X. Tang (China), E. A. Herman, N. Hodges (UK), S. Kaczkowski, O. Kouba (Syria), P. W. Lindstrom, O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, and the proposer.

C10. *Due to Douglas Lind, suggested by the editors.* Show that there are infinitely many numbers that appear at least six times in Pascal's triangle.

How Much of a Parabolic Arc Can Fit in a Unit Disk?

C9. *From the 2001 Putnam Competition.* Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Solution. The answer is yes. For a positive real number A, the parabola $y = Ax^2$ intersects the circle $x^2 + (y-1)^2 = 1$ at the origin and at the points $(\sqrt{2A-1}/A, 2-1/A)$ and $(-\sqrt{2A-1}/A, 2-1/A)$. The length $L(A)$ of the parabolic arc between these points consists of two congruent parts, one in each quadrant. Expressing the length of one of these parts as an integral with respect to the variable y and then letting $u = Ay$, we obtain

$$L(A) = 2\int_0^{2-1/A} \sqrt{1+\frac{1}{4Ay}}\,dy = \frac{2}{A}\int_0^{2A-1} \sqrt{1+\frac{1}{4u}}\,du.$$

It suffices to find a value of A so that $L(A)$ is greater than 4. This occurs when

$$\int_0^{2A-1} \left(\sqrt{1+\frac{1}{4u}} - 1\right) du \geq 1.$$

Since

$$\left(\sqrt{1+\frac{1}{4u}} - 1\right)\left(\sqrt{1+\frac{1}{4u}} + 1\right) = \frac{1}{4u},$$

when $u > 1/12$ we have

$$\sqrt{1+\frac{1}{4u}} - 1 \geq \frac{1}{12u}.$$

Therefore

$$\int_0^{2A-1} \left(\sqrt{1+\frac{1}{4u}} - 1\right) du \geq \int_1^{2A-1} \left(\sqrt{1+\frac{1}{4u}} - 1\right) du \geq \int_1^{2A-1} \frac{1}{12u}\,du.$$

Because $\int_1^\infty (1/x)\,dx$ diverges, we may choose A so large that this last integral exceeds 1.

Editorial comments. Numerical calculation shows that the longest arc is achieved when A is approximately 94.1, at which point the length is approximately 4.00267. The figure shows this longest parabolic arc. Not until A is approximately 37 does the arc length exceed 4.

In the 2001 Putnam Competition, just one participant (out of approximately 3000) earned full credit for solving this problem.