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Problems and Solutions

Edited by Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

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PROBLEMS AND SOLUTIONS

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Proposed problems, solutions, and classics should be submitted online at americanmathematicalmonthly.submittable.com/submit.
Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by May 31, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12363. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* A *knight's path* on an n-by-n chessboard is a sequence of distinct integer pairs $(x_1, y_1), \dots, (x_t, y_t)$ such that $1 \le x_k, y_k \le n$ for $1 \le k \le t$ and $\{|x_{k+1} - x_k|, |y_{k+1} - y_k|\} = \{1, 2\}$ for $1 \le k < t$.
(a) For each $n \ge 1$, how many cells can be covered by such a path, when the knight is restricted to cells (x, y) for which $x \equiv y \pmod 3$?
(b) Same question as **(a)**, but with $x \equiv y - 1 \pmod 3$?

12364. *Proposed by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* Let n be a positive integer, and let z be a complex number not in $\{-1, \dots, -n\}$. Prove

$$\sum_{1\le j\le k\le n} \frac{(-1)^{k-1}}{(z+j)k^2}\binom{z+n}{n-k} = \binom{z+n}{n} \sum_{1\le j\le k\le n} \frac{1}{(z+j)^2 k}.$$

where $\binom{\alpha}{k} = (1/k!)\prod_{i=0}^{k-1}(\alpha - i)$.

12365. *Proposed by Joe Buhler, Reed College, Portland, OR, and George Stoica, Saint John, New Brunswick, Canada.* For positive integers a, b, and c, let $s_k = a^k + b^k + c^k$ for every positive integer k. It is easy to check that if $(a, b, c) = (1, 1, 1)$ or $(a, b, c) = (1, 1, 4)$, then s_k is divisible by s_1 for all k.
(a) Show that if s_1 divides each of s_j, s_{j+1}, and s_{j+2} for some positive integer j, then s_1 also divides s_k for all $k > j + 2$.
(b) For $k > 1$, call a triple (a, b, c) *good for k* if a, b, and c have no common factor greater than 1 and if s_1 divides both s_k and s_{k+1}. Show that $(1, 1, 1)$ and (permutations of) $(1, 1, 4)$ are the only triples that are good for $k = 2$ and also are the only triples that are good for $k = 3$.
(c) Show that if $k \equiv 1 \pmod 3$, then there are infinitely many triples that are good for k.
(d) Show that there are infinitely many values of k for which the number of good triples is finite.

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12366. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Ploieşti, Romania.* **(a)** Find the infimum of

$$\frac{1}{ab+5}+\frac{1}{bc+5}+\frac{1}{cd+5}+\frac{1}{da+5} \tag{$*$}$$

over all nonnegative real numbers a, b, c, and d such that $ab+ac+ad+bc+bd+cd=6$.
(b) Find the infimum of $(*)$ subject to the additional requirement that $a \geq b \geq c \geq d$.

12367. *Proposed by Paolo Perfetti, Tor Vergata University of Rome, Rome, Italy.* Evaluate

$$\sum_{k=0}^{\infty}(k+1)\int_{(2k+1)\pi}^{(2k+3)\pi}\frac{\sin(px)}{x^2+a^2}dx,$$

where p and a are real numbers with $-1 < p < 1$ and $a > 0$.

12368. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* According to problem A3 in the 1970 Putnam Competition, no perfect square can have a decimal representation ending in 4444. There are, however, perfect squares with a decimal representation ending in 444. For $n \geq 4$, how many perfect squares k have a decimal representation that consists of n digits ending in 444?

12369. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let S be the simplex in n-dimensional Euclidean space with vertices $A_0, \ldots, A_n$. Let H_i be the hyperplane containing the vertices of S other than A_i, and let G be the centroid of S. Let P be any point not in any H_i, and let P_i be the point in H_i such that PP_i is parallel to GA_i. Prove that the centroid of the simplex with vertices $P_0, \ldots, P_n$ lies on the line segment PG.

SOLUTIONS

Golden Eigenvalues of Special Matrices

12240 [2021, 276]. *Proposed by Yue Liu, Fuzhou University, Fuzhou, China, and Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL.* We denote by A^* the conjugate transpose of the matrix A.
(a) Let $x \in \mathbb{C}^m$ be a unit column vector. Find the eigenvalues of the $(m+1)$-by-$(m+1)$ matrices

$$\begin{bmatrix} x^*x & x^* \\ x & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} xx^* & x \\ x^* & 0 \end{bmatrix}.$$

(b) More generally, let X be an m-by-n complex matrix, and let ρ be any real number. Find the eigenvalues of the $(m+n)$-by-$(m+n)$ matrices

$$\begin{bmatrix} X^*X & X^* \\ X & \rho I_m \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} XX^* & X \\ X^* & \rho I_n \end{bmatrix}.$$

Solution to part **(a)** *by Jean-Pierre Grivaux, Paris, France.* Let M and N be the two specified matrices. Since x is a unit vector, $x^*x = 1$. The rank of M is two. Thus it has two nonzero eigenvalues λ_1 and λ_2, plus 0 with multiplicity $m-1$. Note $\lambda_1 + \lambda_2 = \operatorname{tr}(M) = 1$. We calculate M^2:

$$M^2 = \begin{bmatrix} 2 & x^* \\ x & xx^* \end{bmatrix}.$$

With the entries of x indexed as $x_1, \ldots, x_m$, the m-by-m matrix xx^* has diagonal entries $|x_1|^2, \ldots, |x_m|^2$. Thus $\operatorname{tr}(M^2) = 2 + \sum |x_i|^2 = 3$, so $\lambda_1^2 + \lambda_2^2 = 3$. Substituting $\lambda_2 = 1 - \lambda_1$ yields a quadratic equation, and we obtain $\{\lambda_1, \lambda_2\} = \{(1-\sqrt{5})/2, (1+\sqrt{5})/2\}$.

The argument for N is similar; it also has rank 2 and trace 1. Now

$$N^2 = \begin{bmatrix} 2xx^* & xx^*x \\ x^*xx^* & 1 \end{bmatrix},$$

so $\operatorname{tr}(N^2) = 3$. Again the two nonzero eigenvalues are $(1-\sqrt{5})/2$ and $(1+\sqrt{5})/2$.

Solution to part **(b)** *by Kuldeep Sarma, Tezpur University, Tezpur, India.* Again let M and N be the two specified matrices. We use the singular value decomposition (SVD). The SVD factors the m-by-n complex matrix X as $U\Sigma V^*$, where U is an m-by-m complex unitary matrix, V is an n-by-n complex unitary matrix, and Σ is an m-by-n rectangular diagonal matrix with nonnegative real numbers $\sigma_1, \ldots, \sigma_s$ on the diagonal, where $s = \min\{m, n\}$. We can then write

$$M = \begin{bmatrix} V\Sigma^*\Sigma V^* & V\Sigma^* U^* \\ U\Sigma V^* & U[\rho I_m]U^* \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \Sigma^*\Sigma & \Sigma^* \\ \Sigma & \rho I_m \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}.$$

Since multiplication by a unitary matrix does not change eigenvalues, it suffices to find the eigenvalues of the matrix S given by

$$S = \begin{bmatrix} \Sigma^*\Sigma & \Sigma^* \\ \Sigma & \rho I_m \end{bmatrix}.$$

We consider a simultaneous permutation of the rows and columns of S, which does not change the eigenvalues. Since Σ is nonzero only on its diagonal, many entries in S are 0. Index the first n rows (and columns) of S as 1 through n, and index the last m rows (and columns) as $1'$ through m'. Let $s = \min\{m, n\}$. Reorder the rows (and columns) in the order $(1, 1', 2, 2', \ldots, s, s')$, followed by the remaining $m + n - 2s$ rows (and columns) in their original order. This converts S to a block-diagonal matrix S' in which the ith block, for $1 \le i \le s$, is the 2-by-2 matrix

$$\begin{bmatrix} \sigma_i^2 & \sigma_i \\ \sigma_i & \rho \end{bmatrix},$$

and the final $m + n - 2s$ blocks are 1-by-1 blocks that are all $[\rho]$ if $m > n$ and are all $[0]$ if $m < n$ (there are none of these 1-by-1 blocks if $m = n$). Note that $m + n - 2s = |m - n|$.

The eigenvalues are the eigenvalues of the blocks: 0 or ρ with the stated multiplicity $|m - n|$, plus

$$\frac{\rho + \sigma_i^2 \pm \sqrt{\left(\rho - \sigma_i^2\right)^2 + 4\sigma_i^2}}{2}$$

from the block for σ_i, where $1 \le i \le s$. Note that if $\sigma_i = 0$, then the block for σ_i reduces to two extra 1-by-1 blocks $[0]$ and $[\rho]$, but this is in fact described by the formula given above for the eigenvalues of the block for σ_i.

The matrix N is generated in the same way as the matrix M, using X^* instead of X. It follows that the spectrum of N is the same as the spectrum of M, except that the multiplicities of 0 and ρ generated by the 1-by-1 blocks are, respectively, $\max\{m - n, 0\}$ and $\max\{n - m, 0\}$, obtained by interchanging the roles of m and n.

Also solved by D. Fleischman, K. Gatesman, L. Han (US) & X. Tang (China), E. A. Herman, C. P. A. Kumar (India), O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, E. I. Verriest, T. Wiandt, and the proposer.

An Integral Limit for This Year—Or, As It Turns Out, Any Year

12242 [2021, 277]. *Proposed by Elena Corobea, Technical College Carol I, Constanţa, Romania.* For $n \geq 1$, let

$$I_n = \int_0^1 \frac{\left(\sum_{k=0}^{n} x^k/(2k+1)\right)^{2022}}{\left(\sum_{k=0}^{n+1} x^k/(2k+1)\right)^{2021}}\,dx.$$

Let $L = \lim_{n\to\infty} I_n$. Compute L and $\lim_{n\to\infty} n(I_n - L)$.

Solution by Kyle Gatesman (student), Johns Hopkins University, Baltimore, MD. We show that $L = 2\ln 2$ and $\lim_{n\to\infty} n(I_n - L) = -1/2$.

For integers $n \geq 1$ and $p \geq 0$, let

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{2k+1} \qquad \text{and} \qquad I_n(p) = \int_0^1 \frac{(S_n(x))^{p+1}}{(S_{n+1}(x))^p}\,dx.$$

For $p \geq 1$,

$$\begin{aligned}
I_n(p) &= \int_0^1 \frac{(S_n(x))^p}{(S_{n+1}(x))^{p-1}} \cdot \frac{S_n(x)}{S_{n+1}(x)}\,dx \\
&= \int_0^1 \frac{(S_n(x))^p}{(S_{n+1}(x))^{p-1}} \cdot \left(1 - \frac{x^{n+1}}{(2n+3)S_{n+1}(x)}\right)dx \\
&= I_n(p-1) - \int_0^1 \left(\frac{S_n(x)}{S_{n+1}(x)}\right)^p \cdot \frac{x^{n+1}}{2n+3}\,dx.
\end{aligned}$$

For $x \in [0, 1]$, we have

$$0 \leq \left(\frac{S_n(x)}{S_{n+1}(x)}\right)^p \cdot \frac{x^{n+1}}{2n+3} \leq \frac{x^{n+1}}{2n+3},$$

so

$$0 \leq I_n(p-1) - I_n(p) \leq \int_0^1 \frac{x^{n+1}}{2n+3}\,dx = \frac{1}{(n+2)(2n+3)}.$$

Therefore $\lim_{n\to\infty}(I_n(p-1) - I_n(p)) = 0$, and by a straightforward induction on p we conclude that $\lim_{n\to\infty}(I_n(0) - I_n(p)) = 0$ for all $p \in \mathbb{Z}^+$. Moreover, for any constant $c \in \mathbb{R}$,

$$0 \leq n(I_n(p-1) - c) - n(I_n(p) - c) \leq \frac{n}{(n+2)(2n+3)},$$

and so $\lim\limits_{n\to\infty}(n(I_n(p-1) - c) - n(I_n(p) - c)) = \lim\limits_{n\to\infty}(n(I_n(0) - c) - n(I_n(p) - c)) = 0$.

Because

$$I_n(0) = \int_0^1 S_n(x)\,dx = \int_0^1 \sum_{k=0}^{n} \frac{x^k}{2k+1}\,dx = \sum_{k=0}^{n} \frac{1}{(k+1)(2k+1)},$$

we conclude

$$\begin{aligned}
\lim_{n\to\infty} I_n(p) &= \lim_{n\to\infty} I_n(0) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{(k+1)(2k+1)} = \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)} \\
&= 2\sum_{k=0}^{\infty} \frac{1}{(2k+2)(2k+1)} = 2\sum_{k=0}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 2\ln 2.
\end{aligned}$$

In particular, in the case $p = 2021$, we obtain $L = 2\ln 2$.

Similarly, observe that

$$\begin{aligned}
\lim_{n\to\infty} n(I_n(p) - L) &= \lim_{n\to\infty} n(I_n(0) - L) \\
&= \lim_{n\to\infty} n\left(\sum_{k=0}^{n} \frac{1}{(k+1)(2k+1)} - \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)}\right) \\
&= \lim_{n\to\infty} n\left(-\sum_{k=n+1}^{\infty} \frac{1}{(k+1)(2k+1)}\right).
\end{aligned}$$

For every $n \in \mathbb{Z}^+$ we have

$$\sum_{k=n+1}^{\infty} \frac{1}{(k+1)(2k+4)} \le \sum_{k=n+1}^{\infty} \frac{1}{(k+1)(2k+1)} \le \sum_{k=n+1}^{\infty} \frac{1}{(k+1)2k}.$$

Since

$$\sum_{k=n+1}^{\infty} \frac{1}{(k+1)(2k+4)} = \frac{1}{2}\sum_{k=n+1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2(n+2)}$$

and

$$\sum_{k=n+1}^{\infty} \frac{1}{(k+1)2k} = \frac{1}{2}\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2(n+1)},$$

we conclude

$$-\frac{n}{2(n+1)} \le n\left(-\sum_{k=n+1}^{\infty} \frac{1}{(k+1)(2k+1)}\right) \le -\frac{n}{2(n+2)}.$$

Thus, by the squeeze theorem,

$$\lim_{n\to\infty} n(I_n(p) - L) = \lim_{n\to\infty} n\left(-\sum_{k=n+1}^{\infty} \frac{1}{(k+1)(2k+1)}\right) = -\frac{1}{2},$$

and setting $p = 2021$ completes the solution of the stated problem.

Editorial comment. The solution shows that the answers are the same if 2021 and 2022 are replaced by p and $p+1$ for any nonnegative integer p. Indeed, since $I_n(p)$ is a decreasing function of p, the answers are the same if 2021 and 2022 are replaced by x and $x+1$ for any nonnegative real number x.

Also solved by K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), D. Fleischman, L. Han (USA) & X. Tang (China), E. A. Herman, N. Hodges (UK), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, J. Yan (China), and the proposer.

A Hyperbolic Integral

12243 [2021, 277]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* For $a > 0$, evaluate

$$\int_0^a \frac{t}{\sinh t\,\sqrt{1 - \operatorname{csch}^2 a \cdot \sinh^2 t}}\,dt.$$

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. Let $I(a)$ be the desired value. First, we observe that

$$1 - \operatorname{csch}^2 a \sinh^2 t = \cosh^2 t\,(1 - \coth^2 a \tanh^2 t).$$

Using this, we obtain

$$I(a) = \int_0^a \frac{t\,dt}{\sinh t\,\sqrt{1-\operatorname{csch}^2 a\cdot\sinh^2 t}} = \int_0^a \frac{t\,dt}{\sinh t\cosh t\,\sqrt{1-\coth^2 a\cdot\tanh^2 t}}.$$

Now using the substitution $\cos x = \coth a\tanh t$, we have

$$I(a) = \int_0^{\pi/2} \frac{\tanh^{-1}(\tanh a\cos x)}{\cos x}\,dx$$

and hence

$$I'(a) = \int_0^{\pi/2} \frac{\operatorname{sech}^2 a}{1-\tanh^2 a\cos^2 x}\,dx = \operatorname{sech} a\tan^{-1}(\cosh a\tan x)\Big|_0^{\pi/2} = \frac{\pi}{2}\operatorname{sech} a.$$

Thus

$$I(a) = \int_0^a I'(s)\,ds = \frac{\pi}{2}\int_0^a \operatorname{sech} s\,ds = \frac{\pi}{2}\tan^{-1}(\sinh a).$$

Editorial comment. Several solvers noted that the requested integral can be reduced to integral (3.535) from I. S. Gradshteyn, I. M. Ryzhik, et al. (2014), *Table of Integrals, Series, and Products*, 8th edition, Cambridge, MA: Academic Press.

Also solved by U. Abel & V. Kushnirevych (Germany), P. Bracken, H. Chen, G. Fera (Italy), L. Han (US) & X. Tang (China), N. Hodges (UK), O. P. Lossers (Netherlands), T. M. Mazzoli (Austria), M. Omarjee (France), A. Stadler (Switzerland), S. M. Stewart (Saudi Arabia), R. Tauraso (Italy), UM6P Math Club (Morocco), and the proposer.

Equitable Polyominos in a Box

12244 [2021, 376]. *Proposed by Rob Pratt, SAS Institute Inc., Cary, NC, Stan Wagon, Macalester College, St. Paul, MN, Douglas B. West, University of Illinois, Urbana, IL, and Piotr Zielinski, Cambridge, MA.* A *polyomino* is a region in the plane with connected interior that is the union of a finite number of squares from a grid of unit squares. For which integers k and n with $4 \le k \le n$ does there exist a polyomino P contained entirely within an n-by-n grid such that P contains exactly k unit squares in every row and every column of the grid? Clearly such polyominos do not exist when $k = 1$ and $n \ge 2$. Nikolai Beluhov noticed that they do not exist when $k = 2$ and $n \ge 3$, and his Problem 12137 [2019, 756; 2021, 381] shows that they do not exist when $k = 3$ and $n \ge 5$.

Solution by Jacob Boswell, Missouri Southern State University, Joplin, MO. Polyominos with the desired properties, which we call (k, n)*-equitable polyominos*, exist whenever $4 \le k \le n$.

Denote the n-by-n grid by $\mathcal{G}_n$. We call its unit squares *cells* and specify their positions in matrix notation. We call the three cells (1,1), (1,2), and (2,1) the top left *guard*. Similarly, we define top right, bottom left, and bottom right guards.

We argue by induction on k that in $\mathcal{G}_n$ there is a (k, n)-equitable polynomino that contains two diagonally opposite guards such that removing the corner square from one of those guards leaves the remainder connected. Let $\mathcal{C}_{k,n}$ denote the class of such polyominos. We postpone the discussion of the base cases.

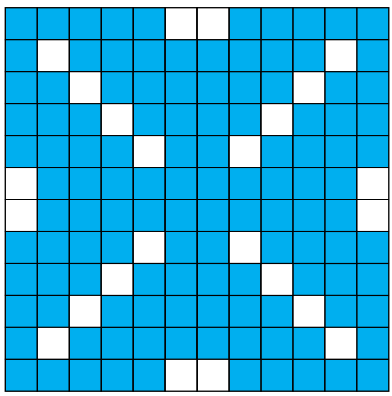
For the induction step, consider (k, n) with $n \ge k \ge 9$. Cover $\mathcal{G}_n$ using two diagonally opposite copies of $\mathcal{G}_{\lceil n/2\rceil}$ and two diagonally opposite copies of $\mathcal{G}_{\lfloor n/2\rfloor}$. When n is odd, the two larger subgrids share one cell in the center, but other than that the subgrids share no cells.

We describe a uniform construction for all cases except when n is odd and k is even. In the two opposite copies of $\mathcal{G}_{\lceil n/2\rceil}$, place members of $\mathcal{C}_{\lceil k/2\rceil,\lceil n/2\rceil}$, with one of the guards that are inductively guaranteed to exist placed in the center of $\mathcal{G}_n$. In the two opposite copies of $\mathcal{G}_{\lfloor n/2 \rfloor}$, similarly place members of $\mathcal{C}_{\lfloor k/2 \rfloor, \lfloor n/2 \rfloor}$ with their guaranteed guards in the center of $\mathcal{G}_n$.

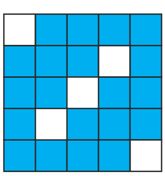
When n is odd and k is even, use members of $\mathcal{C}_{k/2+1, \lceil n/2 \rceil}$ in the larger subgrids and $\mathcal{C}_{k/2-1, \lfloor n/2 \rfloor}$ in the smaller subgrids, and (in this case) delete the central cell from the resulting polyomino. The use of $\mathcal{C}_{k/2-1, \lfloor n/2 \rfloor}$ here is the reason we need $k = 8$ in the basis.

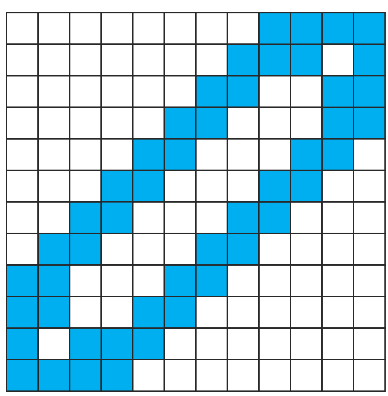
In each case, the guards from each subpolyomino retain a cell adjacent to a cell retained from the guard in a neighboring subpolyomino, so the resulting full polyomino is connected. The polyomino also retains diagonally opposite complete guards, and deleting the corner cell from one of those guards does not disconnect the polyomino, because it does not disconnect the subpolyomino (even when the central cell is deleted, the two neighbors of the central cell are connected through the other subpolyominos).

When n is even, the number of cells in each row and column of the final polyomino is $\lceil k/2 \rceil + \lfloor k/2 \rfloor$. When n is odd and k is odd, the computation is the same except for the central row and column, where it is $\lceil k/2 \rceil + \lceil k/2 \rceil - 1$ as desired, since the central cell contributes only once. When n is odd and k is even, we have $k/2 + 1 + k/2 - 1$ cells in each noncentral row and column, and in the central row and column we have $k/2 + 1 + k/2 + 1 - 2$ cells, since the central cell was deleted. (Keeping the larger subgrid connected in this case is the reason for the special condition on the subgrid.) Below we show the construction of a member of $\mathcal{C}_{10,12}$ from four members of $\mathcal{C}_{5,6}$.



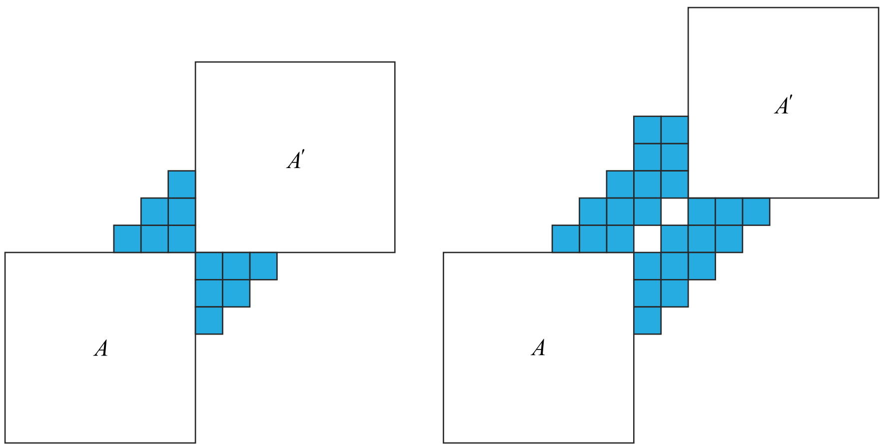
Now we return to the base cases. Because the induction step for k needs the induction hypothesis for $\lfloor (k-1)/2 \rfloor$ and (k, n)-equitable polyominos do not generally exist when $k \le 3$, we need base cases for $4 \le k \le 8$. Below we show members of $\mathcal{C}_{4,5}$ and $\mathcal{C}_{4,12}$. The general construction shown for $(k, n) = (4, 12)$ is valid when $n \ge 6$, which completes the proof for $k = 4$.



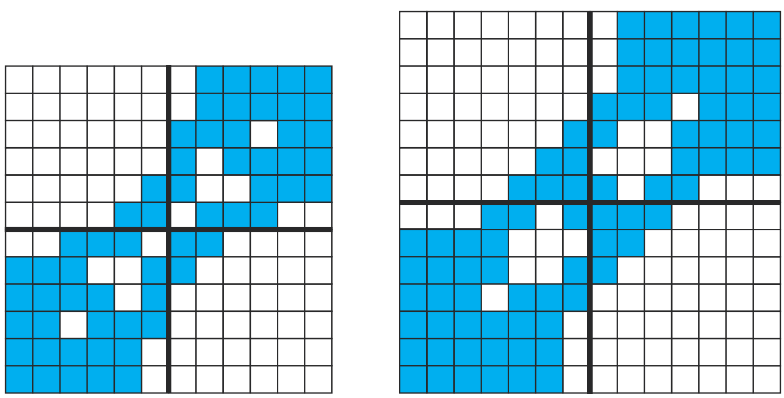


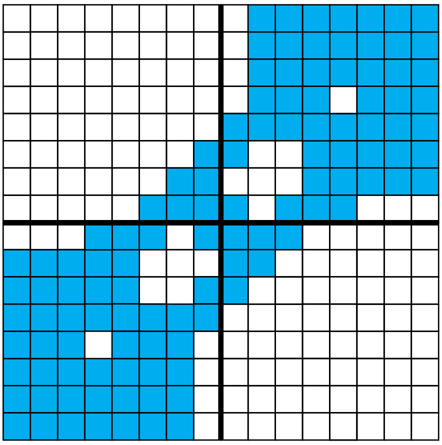
For $k \geq 5$, we show first that a special construction for $n = 2k + 2$ yields constructions for all larger n. Say that a member of $\mathcal{C}_{k,2k+2}$ is a *butterfly* if its portion in the upper left and lower right quadrants consists precisely of triangular arrays of cells with side-length $\lfloor k/2 \rfloor$ touching the center of $\mathcal{G}_{2k+2}$, as indicated on the left below. Suppose that $\mathcal{C}_{k,2k+2}$ contains a butterfly B_k. Note that the polyomino A' in the upper right quadrant of B_k can be assumed to be the transpose of A.

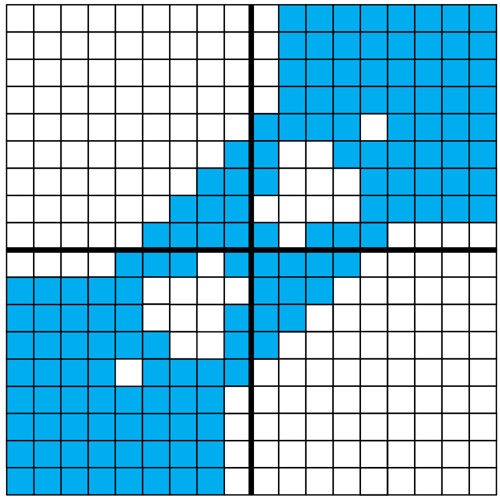
From B_k one can obtain a member of $\mathcal{C}_{k,n}$ whenever $n > 2k + 2$ by enlarging the central portion of the butterfly and spreading A and A' farther apart, as shown on the right below. When k is even, the central diagonal of the added portion is omitted, but when k is odd it is present. The correct counts in the rows and columns occupied by A and A' are inherited from B_k.



Below we show butterflies for $5 \leq k \leq 8$. One issue in these constructions is ensuring that the polyomino is connected; this is the reason we provided a different construction for $k = 4$.







At this point the proof is completed by exhibiting explicit examples for $k \leq n \leq 2k+1$ when $5 \leq k \leq 8$. General constructions for $n = k$ and $n = k+1$ are trivial. What remains is a finite problem, exhibiting 26 polynominos. We leave the constructions to the reader.

Editorial comment. The constructions are far from unique. For example, there is a construction similar to the butterfly that exists when $n = 2k$ and expands like the butterfly, reducing the finite problem to 18 polyominos.

Also solved by K. Gatesman, R. Stong, and the proposer.

CLASSICS

C11. *Suggested by Richard Stanley, University of Miami, Coral Gables, FL.* A standard deck of cards has 26 red cards and 26 black cards. Deal out the cards in a shuffled standard deck, one card at a time. At any point before the last card is dealt, you can guess that the next card is red. For example, you may guess that the very first card is red, and your guess will be correct with probability $1/2$. Or you may watch some cards go by, noting their color in order to decide when to guess. What strategy maximizes the probability that your guess is correct?

Repetitions in the Interior of Pascal's Triangle

C10. *Due to Douglas Lind, suggested by the editors.* Show that there are infinitely many numbers that appear at least six times in Pascal's triangle.

Solution. For $m \geq 3$, m occurs twice as $\binom{m}{1}$ and $\binom{m}{m-1}$. By symmetry, it will suffice to find infinitely many values of m with at least two more occurrences in the left half of the triangle.

There are several small examples of such pairs of occurrences: $120 = \binom{10}{3} = \binom{16}{2}$, $210 = \binom{10}{4} = \binom{21}{2}$, $1540 = \binom{22}{3} = \binom{56}{2}$, and $3003 = \binom{15}{5} = \binom{14}{6}$. The last of these exhibits the intriguing relationship $\binom{n}{k} = \binom{n-1}{k+1}$. To solve the problem, we will find infinitely many solutions of this equation with $k > 1$ and $k+1 < (n-1)/2$.

The equation $\binom{n}{k} = \binom{n-1}{k+1}$ is equivalent to $n(k+1) - (n-k)(n-k-1) = 0$. We claim that for every positive integer j, this equation is satisfied by the values $n = F_{2j+2}F_{2j+3}$

and $k = F_{2j}F_{2j+3}$, where F_i is the ith Fibonacci number. To see why, note that with these values we have $n - k = (F_{2j+2} - F_{2j})F_{2j+3} = F_{2j+1}F_{2j+3}$, and therefore

$$\begin{aligned} n(k+1) - (n-k)(n-k-1) &= F_{2j+2}F_{2j+3}(F_{2j}F_{2j+3} + 1) - F_{2j+1}F_{2j+3}(F_{2j+1}F_{2j+3} - 1) \\ &= F_{2j+3}(F_{2j+2}F_{2j}F_{2j+3} + F_{2j+2} - F_{2j+1}^2 F_{2j+3} + F_{2j+1}) \\ &= F_{2j+3}(F_{2j+2}F_{2j}F_{2j+3} - F_{2j+1}^2 F_{2j+3} + F_{2j+3}) \\ &= F_{2j+3}^2(F_{2j+2}F_{2j} - F_{2j+1}^2 + 1) = 0, \end{aligned}$$

where the last step uses the well-known identity $F_{i+1}F_{i-1} - F_i^2 = (-1)^i$.

The case $j = 1$ yields $n = 15$ and $k = 5$, the example we found earlier. When $j = 2$ we get $n = 104$ and $k = 39$, and indeed $\binom{104}{39} = \binom{103}{40} = 61218182743304701891431482520$.

Editorial comments. The appearance of the Fibonacci numbers in this solution can be explained by reference to classic problem C2 (this MONTHLY, Feb. 2022, p. 194). Viewing the equation $n(k+1) - (n-k)(n-k-1) = 0$ as a quadratic in n and applying the quadratic formula yields

$$n = \frac{3k + 2 \pm \sqrt{5k^2 + 8k + 4}}{2}.$$

For n to be an integer, we need $5k^2 + 8k + 4$ to be a perfect square. Setting $5k^2 + 8k + 4 = t^2$ and solving for k by the quadratic formula, we get

$$k = \frac{-4 \pm \sqrt{5t^2 - 4}}{5}.$$

For k to be an integer, $5t^2 - 4$ must be a perfect square, and the solution to classic problem C2 (March 2022, pp. 293–294) shows that this happens if and only if t is an odd-indexed Fibonacci number. Setting $t = F_{2i+1}$ and applying Fibonacci identities leads to the values

$$n = F_{i+1}F_{i+2} + \frac{(-1)^{i+1} - 1}{5}, \quad k = F_{i-1}F_{i+2} + \frac{4((-1)^{i+1} - 1)}{5}.$$

These are integers when i is odd, and setting $i = 2j + 1$ leads to the values used in the solution.

This result is due to Lind (D. Lind, The quadratic field $Q(\sqrt{5})$ and a certain Diophantine equation, *Fib. Quart.* **6** (1968) 86–94, fq.math.ca/Scanned/6-3/lind.pdf). See also C. A. Tovey, Multiple occurrences of binomial coefficients, *Fib. Quart.* **23** (1985) 356–358. It is related to a 1971 conjecture of Singmaster (D. Singmaster, How often does an integer occur as a binomial coefficient?, this MONTHLY **78** (1971) 385–386). For an integer m with $m \geq 2$, let S_m be the number of times m appears in Pascal's triangle. Singmaster conjectured that S_m is bounded, and suggested that 10 or 12 might be a bound. The problem shows that 5 cannot be an asymptotic bound. It turns out that $S_{3003} = 8$; there are no other known values of m for which $S_m \geq 8$. The sequence of binomial coefficients for which $S_m \geq 6$ starts 120, 210, 1540, 3003, 7140, 11628, 24310, 61218182743304701891431482520 (see the OEIS sequences: oeis.org/A003015, oeis.org/A003016, and oeis.org/A090162). See also K. Matomäki, M. Radziwiłł, X. Shao, T. Tao, and J. Teräväinen, Singmaster's conjecture in the interior of Pascal's triangle, arxiv.org/abs/2106.03335.