The American Mathematical Monthly

ISSN: (Print) (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at americanmathematicalmonthly.submittable.com/submit. *Proposed solutions to the problems below must be submitted by October 31, 2022. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

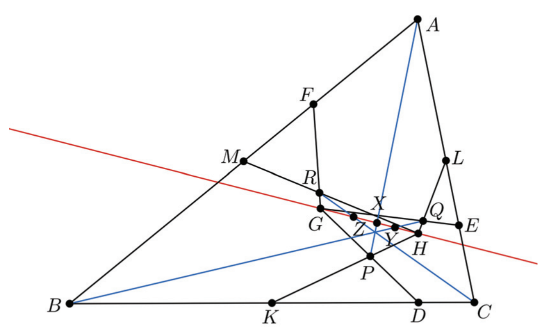
PROBLEMS

12328. *Proposed by Peter Koymans and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.* An integer binary quadratic form is a function $f : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $f(m,n) = am^2 + bmn + cn^2$ for some $a, b, c \in \mathbb{Z}$. The *value set V(f)* of such a form is defined to be $\{f(m,n) : (m,n) \in \mathbb{Z}^2\}$.
(a) Prove that if $f_1(m,n) = m^2 - mn - 3n^2$ and $f_2(m,n) = m^2 - 13n^2$, then $V(f_1) = V(f_2)$.
(b) Prove that if $f_1(m,n) = m^2 - mn - 4n^2$ and $f_2(m,n) = m^2 - 17n^2$, then $V(f_2) \subseteq V(f_1)$ but $V(f_1) \neq V(f_2)$.

12329. *Proposed by Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let n be a positive integer with $n \geq 3$. For each positive integer m with $m \geq 2$, find all real values λ_m such that there are m distinct unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$ satisfying $v_i \cdot v_j = \lambda_m$ for all i, j with $1 \leq i < j \leq m$.

12330. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* In the acute and scalene triangle ABC, let G be the centroid, H be the orthocenter, D, E, and F be the feet of the altitudes from A, B, and C, respectively, and K, L, and M be the midpoints of BC, CA, and AB, respectively. Let P be the intersection of DG and KH, let Q be the intersection of EG and LH, and let R be the intersection of FG and MH.
(a) Prove that AP, BQ, and CR are concurrent.
(b) Let X, Y, and Z be the points where GH intersects AP, BQ, and CR. Prove

$$\frac{HX}{XG} + \frac{HY}{YG} + \frac{HZ}{ZG} = 3.$$



http://dx.doi.org/doi.org/10.1080/00029890.2022.2051930

12331. *Proposed by WeChat Group on Matrix Analysis, Nova Southeastern University, Fort Lauderdale, FL.* Let A and B be complex m-by-n matrices, and let C be a complex n-by-m matrix. Prove that if there are nonzero scalars x and y such that $ACB = xA + yB$, then $ACB = BCA$.

12332. *Proposed by Finbarr Holland, University College, Cork, Ireland.* Prove

$$\int_0^\infty \frac{\tanh^2 x}{x^2}\,dx = \frac{14\,\zeta(3)}{\pi^2},$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^\infty 1/k^3$.

12333. *Proposed by Moshe Rosenfeld, University of Washington, Seattle, WA, and Tacoma Institute of Technology, Tacoma, WA.* Let G be the multigraph obtained by replacing each edge of the complete graph K_{12} by five edges. Show that the 330 edges of G can be partitioned into 11 sets such that each set forms a graph isomorphic to the icosahedron.

12334. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.* Let f be a real-valued function on $[0, 1]$ with a continuous second derivative. Assume that $f(0) = 0$, $f'(0) = 1$, $f''(0) \neq 0$, and $0 < f'(x) < 1$ for all $x \in (0, 1]$. Let $x_1, x_2, \ldots$ be a sequence with $0 < x_1 \leq 1$ and with

$$x_n = f\left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)$$

for $n \geq 2$. Prove $\lim_{n\to\infty} x_n \ln n = -2/f''(0)$.

SOLUTIONS

Dominated Convergence of an Integral

12207 [2020, 753]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $\int_0^1 f(x)\,dx = 1$. Evaluate

$$\lim_{n\to\infty} \frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1 - x)\,dx.$$

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. Substituting $t = x^n$, we get

$$\frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1 - x)\,dx = -\int_0^1 f(t)u_n(t)\,dt,$$

where

$$u_n(t) = -\frac{t^{1/n} \ln(1 - t^{1/n})}{\ln n}.$$

For fixed $t \in (0, 1)$, letting $y = 1/n$ and applying L'Hôpital's rule twice yields

$$\lim_{n\to\infty} u_n(t) = \lim_{y\to 0^+} \frac{\ln(1 - t^y)}{\ln y} = \lim_{y\to 0^+} \frac{t^y \ln t/(t^y - 1)}{1/y} = \lim_{y\to 0^+} \frac{y \ln t}{t^y - 1} = \lim_{y\to 0^+} \frac{\ln t}{t^y \ln t} = 1.$$

Moreover, by Bernoulli's inequality, for $n \geq 3$ we have

$$0 \leq u_n(t) \leq -\frac{\ln(1 - t^{1/n})}{\ln n} \leq -\frac{\ln((1 - t)/n)}{\ln n} = 1 - \frac{\ln(1 - t)}{\ln n} \leq 1 - \ln(1 - t).$$

Since f is bounded and $\int_0^1 (1 - \ln(1-t))\,dt = 2 < \infty$, the dominated convergence theorem applies, and we conclude that

$$\lim_{n\to\infty} \frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1-x)\,dx = -\lim_{n\to\infty} \int_0^1 f(t) u_n(t)\,dt = -\int_0^1 f(t)\,dt = -1.$$

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), C. Antoni (Italy), R. Boukharfane (Saudi Arabia), N. Caro (Brazil), R. Gordon, N. Grivaux (France), L. Han (USA) & X. Tang (China), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), E. J. Ionaşcu, Y. Jinhai, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), A. Stadler (Switzerland), R. Stong, T. Wilde (UK), Y. Xiang (China), and the proposer.

Three Wise Women

12208 [2020, 753]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* (In memory of John Horton Conway, 1937–2020.) Three wise women, Alice, Beth, and Cecily, sit around a table. A card with a positive integer on it is attached to each woman's forehead, so she can see the other two numbers but not her own. The women know that one of the three integers is equal to the sum of the other two. The same question, "Can you determine the number on your forehead?", is addressed to the wise women in the following order: Alice, Beth, Cecily, Alice, Beth, Cecily, ... The answer is either "No" or "Yes, the number is ___," and the other wise women hear the answer. The questioning ends as soon as the positive answer is obtained. (Assume that the women are logical and honest, they all know this, they all know that they all know this, and so on.)

(a) Prove that whichever numbers are assigned to the wise women, an affirmative answer is obtained eventually.

(b) Suppose that Alice's second answer is "Yes, the number is 50." Determine the numbers assigned to Beth and Cecily.

(c) Suppose the numbers assigned to Alice, Beth, and Cecily are 1492, 1776, and 284, respectively. Determine who will give the affirmative answer and how many negative answers she will give before that.

Solution by Mark D. Meyerson, US Naval Academy, Annapolis, MD. We describe each assignment of numbers with a triple (a, b, c) giving Alice's, Beth's, and Cicely's positive numbers in that order. Note that one of the entries must be the sum of the other two.

We claim that for all triples, if a woman says "Yes" on some turn, then her number must be the largest. Suppose not, and choose a counterexample (a, b, c) for which the "Yes" answer occurs as early as possible. Suppose, for example, Alice says "Yes" on turn n, but Beth has the largest number, so $b = a + c$. (Other cases are similar.) Alice, seeing the numbers $a + c$ and c, knows from the beginning that her number must be either a or $a + 2c$. To say "Yes" on turn n, she must be able to rule out the triple $(a + 2c, a + c, c)$ for the first time on that turn, and this will happen only if either Cicely or Beth would have said "Yes" on turn $n - 1$ or $n - 2$ on that triple. But this is ruled out by the minimality of n, since neither Beth nor Cicely has the largest number in that triple.

Let f be the function that assigns to a triple the number of the turn on which the answer "Yes" occurs. Part (**a**) asks us to show that f is defined for every triple. If the triple has the form $(2x, x, x)$, for some positive integer x, then Alice will say "Yes" on her first turn, so $f(2x, x, x) = 1$. If it has the form $(x, 2x, x)$, then Alice will think she could have either x or $3x$, so she will say "No," and then Beth will say "Yes." Therefore $f(x, 2x, x) = 2$. Similarly, for triples of the form $(x, x, 2x)$, Cicely will say "Yes" on her first turn, and $f(x, x, 2x) = 3$.

Now consider triples in which the numbers are distinct. If some triple never yields an affirmative answer, then let (a, b, c) be such a triple whose largest element is as small as possible. If $c = a + b$, then $(a, b, |a - b|)$ has a smaller largest element, so $f(a, b, |a - b|)$ is defined. If $f(a, b, |a - b|) = n$, then on turn $n + 1$ or $n + 2$, depending on which of a or b is larger, Cecily can eliminate the triple $(a, b, |a - b|)$, since Alice or Beth would previously have said "Yes." Cecily then answers "Yes" with $a + b$ on her turn. The argument is similar when a or b is the largest entry in (a, b, c). This completes the solution to (**a**).

(**b**) Using the reasoning from part (**a**), we can now determine, for every n, the triples (a, b, c) for which $f(a, b, c) = n$. If $f(a, b, c) = 1$, then (a, b, c) must have the form $(2x, x, x)$, for some positive integer x. For $f(a, b, c) = 2$, we must have $b = a + c$. If $a = c$ then (a, b, c) has the form $(x, 2x, x)$. If not, then $f(a, |a - c|, c)$ must be 1, so $(a, |a - c|, c)$ has the form $(2x, x, x)$, and therefore $(a, b, c) = (2x, 3x, x)$. Thus, the triples (a, b, c) such that $f(a, b, c) = 2$ are those of the form $(x, 2x, x)$ or $(2x, 3x, x)$. If $f(a, b, c) = 3$, then $c = a + b$, and either (a, b, c) has the form $(x, x, 2x)$ or $f(a, b, |a - b|)$ is either 1 or 2, in which case (a, b, c) has the form $(2x, x, 3x)$, $(x, 2x, 3x)$, or $(2x, 3x, 5x)$. A similar argument shows that the triples (a, b, c) with $f(a, b, c) = 4$ are those of the form $(3x, 2x, x)$, $(4x, 3x, x)$, $(3x, x, 2x)$, $(4x, x, 3x)$, $(5x, 2x, 3x)$, or $(8x, 3x, 5x)$. Since 50 is not divisible by any number in $\{3, 4, 8\}$, the only way Alice will say "Yes, my number is 50" on her second turn ($n = 4$) is for x to be 10 in the fifth triple, so Beth has 20 and Cecily has 30.

(c) Working from $(1492, 1776, 284)$ to determine the turn on which that triple will be resolved, we iteratively replace the biggest number by the difference of the other two to undo the decision process. The successive triples after $(1492, 1776, 284)$ are these: $(1492, 1208, 284)$, $(924, 1208, 284)$, $(924, 640, 284)$, $(356, 640, 284)$, $(356, 72, 284)$, $(212, 72, 284)$, $(212, 72, 140)$, $(68, 72, 140)$, $(68, 72, 4)$, $(68, 64, 4)$, $(60, 64, 4)$, $(60, 56, 4)$, $(52, 56, 4)$, $(52, 48, 4)$, $(44, 48, 4)$, $(44, 40, 4)$, $(36, 40, 4)$, $(36, 32, 4)$, $(28, 32, 4)$, $(28, 24, 4)$, $(20, 24, 4)$, $(20, 16, 4)$, $(12, 16, 4)$, $(12, 8, 4)$, $(4, 8, 4)$. The last triple would be resolved by Beth on turn 2, the one before it by Alice on turn 4. Working backward, Yes comes on the following turns for these triples:

$$2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 27, 28, 30, 31, 32, 34, 35, 37, 38.$$

Since $38 = 3 \cdot 12 + 2$, the affirmative answer is by Beth after giving 12 negative answers. (Tracking only the two smaller entries in each triple, the decision process parallels the Euclidean algorithm.)

Also solved by E. Curtin, J. Boswell & C. Curtis, N. Hodges (UK), E. J. Ionaşcu, G. Lavau (France), O. P. Lossers (Netherlands), K. Schilling, E. Schmeichel, R. Stong, F. A. Velandia & J. F. Gonzalez (Colombia), T. Wilde (UK), Eagle Problem Solvers, The Zurich Logic Coffee (Switzerland), and the proposer.

Asymptotics of a Recursively Defined Sequence

12210 [2020, 852]. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.* Let $x_1 = 1$, and let

$$x_{n+1} = \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right)^2$$

when $n \geq 1$. For $n \in \mathbb{N}$, let $a_n = 2n + (1/2)\log n - x_n$. Show that the sequence $a_1, a_2, \ldots$ converges.

Solution by Peter W. Lindstrom, Saint Anselm College, Manchester, NH. By the recurrence for x_n, we have $x_{n+1} = x_n + 2 + 1/x_n > x_n + 2$, and therefore by induction $x_n \geq 2n$ when $n > 1$.

Let $z_k = x_k - 2k$. Since $z_{k+1} - z_k = x_{k+1} - x_k - 2 = 1/x_k$, we have

$$z_n = z_1 + \sum_{k=1}^{n-1}(z_{k+1} - z_k) = -1 + \sum_{k=1}^{n-1} \frac{1}{x_k}$$

for $n > 1$. Thus

$$0 \le \frac{1}{2k} - \frac{1}{x_k} = \frac{z_k}{2kx_k} = \frac{\sum_{j=1}^{k-1} 1/x_j - 1}{2kx_k} \le \frac{\sum_{j=2}^{k-1} 1/x_j}{(2k)^2} \le \frac{(1/2)\sum_{j=2}^{k-1} 1/j}{4k^2} < \frac{\log k}{8k^2}$$

for $k > 2$. Since $\sum_{k=1}^{\infty} \log k/(8k^2)$ is convergent, so is $\sum_{k=1}^{\infty}(1/(2k) - 1/x_k)$. Let

$$\zeta = \sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{x_k}\right).$$

For $n > 1$,

$$\begin{aligned} a_n = 2n + \frac{\log n}{2} - x_n = -z_n + \frac{\log n}{2} &= 1 - \sum_{k=1}^{n-1} \frac{1}{x_k} + \frac{\log n}{2} \\ &= 1 - \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) + \sum_{k=1}^{n-1}\left(\frac{1}{2k} - \frac{1}{x_k}\right) + \frac{1}{2n}. \end{aligned}$$

Thus $\lim_{n\to\infty} a_n = 1 - \gamma/2 + \zeta$, where γ is the Euler–Mascheroni constant.

Also solved by G. Aggarwal (India), K. F. Andersen (Canada), M. Bataille (France), R. Boukharfane (Saudi Arabia), H. Chen, C. Chiser (Romania), Ó. Ciaurri (Spain), C. Degenkolb, A. Dixit (India) & S. Pathak (USA), G. Fera (Italy), J. Freeman (Netherlands), R. Gordon, J.-P. Grivaux (France), L. Han, R. Hang, D. Henderson, E. A. Herman, N. Hodges (UK), Y. Jinhai (China), O. Kouba (Syria), Z. Lin (China), J. H. Lindsey II, O. P. Lossers (Netherlands), S. Omar (Morocco), M. Omarjee (France), P. Palmieri & C. Antoni (Italy), A. Pathak (India), R. K. Schwartz, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, E. I. Verriest, J. Vukmirović (Serbia), T. Wiandt, L. Wimmer (Germany), L. Zhou, and the proposer.

A Truncated Tetrahedron

12211 [2020, 852]. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania.* On each of the six edges of a tetrahedron, identify the point that is coplanar with the incenter of the tetrahedron and with the two vertices incident to the opposite edge. Prove that the volume of the octahedron formed by these six points is no more than half the volume of the tetrahedron, and determine the conditions for equality.

Solution by Elton Bojaxhiu, Tirana, Albania, and Enkel Hysnelaj, Sydney, Australia. Let A, B, C, and D be the vertices of the tetrahedron, and let w, x, y, and z denote the areas of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$, respectively.

Let p_{AB} be the plane passing through C, D, and the incenter of the tetrahedron, and let P_{AB} denote the intersection of p_{AB} with AB. Let h_A and h_B be the altitudes from A and B, respectively, to the line CD, and let d_A and d_B be the distances from A and B, respectively, to the plane p_{AB}. Since p_{AB} bisects the angle between the planes containing $\triangle ACD$ and $\triangle BCD$, we have

$$\frac{AP_{AB}}{BP_{AB}} = \frac{d_A}{d_B} = \frac{h_A}{h_B} = \frac{y}{z}.$$

Similarly, if P_{AC}, P_{AD}, P_{BC}, P_{BD}, and P_{CD} are the vertices of the octahedron that lie on the other edges of the tetrahedron, then we have

$$\frac{AP_{AC}}{CP_{AC}} = \frac{x}{z}, \quad \frac{AP_{AD}}{DP_{AD}} = \frac{w}{z}, \quad \frac{BP_{BC}}{CP_{BC}} = \frac{x}{y}, \quad \frac{BP_{BD}}{DP_{BD}} = \frac{w}{y}, \quad \text{and} \quad \frac{CP_{CD}}{DP_{CD}} = \frac{w}{x}.$$

The octahedron is constructed from the tetrahedron $ABCD$ by removing the four smaller tetrahedra $AP_{AB}P_{AC}P_{AD}$, $BP_{AB}P_{BC}P_{BD}$, $CP_{AC}P_{BC}P_{CD}$, and $DP_{AD}P_{BD}P_{CD}$. If t is the volume of the tetrahedron $ABCD$ and t_A is the volume of $AP_{AB}P_{AC}P_{AD}$, then

$$\frac{t_A}{t} = \frac{AP_{AD}}{AD} \cdot \frac{AP_{AC}}{AC} \cdot \frac{AP_{AB}}{AB} = \frac{w}{w+z} \cdot \frac{x}{x+z} \cdot \frac{y}{y+z}.$$

Combining this with similar formulas for the other small tetrahedra, we see that it suffices to show

$$\frac{wxy}{(w+z)(x+z)(y+z)} + \frac{wxz}{(w+y)(x+y)(z+y)} + \frac{wyz}{(w+x)(y+x)(z+x)} + \frac{xyz}{(x+w)(y+w)(z+w)} \ge \frac{1}{2}. \quad (*)$$

Let a, b, c, and d denote the elementary symmetric polynomials in w, x, y, and z:

$$\begin{aligned} a &= w + x + y + z, \\ b &= wx + wy + wz + xy + xz + yz, \\ c &= wxy + wxz + wyz + xyz, \\ d &= wxyz. \end{aligned}$$

By multiplying out and rearranging, we find that $(*)$ is equivalent to

$$abc - 5a^2d \ge c^2.$$

From Newton's inequalities for the elementary symmetric polynomials, we have $(a/4)(c/4) \le (b/6)^2$ and $(b/6)d \le (c/4)^2$. Consequently,

$$b \ge \frac{3\sqrt{ac}}{2} \quad \text{and} \quad d \le \frac{3c^2}{8b} \le \frac{3c^2}{12\sqrt{ac}} = \frac{c^{3/2}}{4\sqrt{a}}.$$

Also, by Maclaurin's inequality, $a/4 \ge \sqrt[3]{c/4}$, so $a^{3/2} \ge 4\sqrt{c}$. Therefore

$$abc - 5a^2d \ge ac \cdot \frac{3\sqrt{ac}}{2} - 5a^2 \cdot \frac{c^{3/2}}{4\sqrt{a}} = \frac{a^{3/2}c^{3/2}}{4} \ge \frac{4\sqrt{c} \cdot c^{3/2}}{4} = c^2,$$

as required.

Equality holds if and only if $w = x = y = z$; that is, all faces of the tetrahedron have the same area. It is well known that this is true precisely when the tetrahedron is *isosceles*, which means that each pair of opposite edges have the same length.

Editorial comment. There are several other ways to establish $(*)$, as indicated by multiple solvers. For instance, one could cite Muirhead's inequality; alternatively, assume without loss of generality that $w \le x \le y \le z$, write $x = w + s$, $y = w + s + t$, and $z = w + s + t + u$ for $s, t, u \ge 0$, and note that expanding and rearranging $(*)$ yields $f(w, s, t, u) \ge 0$, where f is a polynomial with all nonnegative coefficients.

Also solved by C. Curtis, G. Fera (Italy), O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, J. Vukmirović, and the proposer.

An Application of Farkas's Lemma

12212 [2020, 852]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $x_1, \dots, x_m$ and $y_1, \dots, y_m$ be two lists of m vectors in $\mathbb{R}^n$, and suppose

$$\langle x_i - x_j, y_i - y_j \rangle \ge 0$$

for all i and j in $\{1, \dots, m\}$. Prove that there exists a vector y in $\mathbb{R}^n$ such that

$$\langle x_i, y_i \rangle \ge \langle x_i, y \rangle$$

for all i in $\{1, \dots, m\}$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. The following is a variant of Farkas's lemma (see for example Corollary 7.1(e) in A. J. Schrijver, *Theory of Linear and Integer Programming*, John Wiley and Sons, Chichester, UK, 1986).

If A is a p-by-q matrix, and $b \in \mathbb{R}^p$, then exactly one of the following two assertions is true:

(1) The system $Au \leq b$ has a solution $u \in \mathbb{R}^q$.

(2) The system $v^T A = 0$ has a solution $v \in \mathbb{R}^p$ with $v \geq 0$ and $v^T b < 0$.

Let X and Y be the n-by-m matrices that have the vectors x_i and y_i, respectively, for their columns. Let $A = X^T Y$; in particular, the (i, j)-entry of A is $\langle x_i, y_j \rangle$. Let b be the vector consisting of the main diagonal entries of A. If some vector u satisfies $Au \leq b$, then the vector y defined by

$$y = Yu = \sum_{j=1}^{m} u_j y_j$$

has the desired property, because

$$\langle x_i, y \rangle = \sum_j u_j \langle x_i, y_j \rangle = \sum_j u_j a_{i,j} = (Au)_i \leq b_i = \langle x_i, y_i \rangle.$$

If there is no such vector u, then by the variant of Farkas's lemma there exists $v \in \mathbb{R}^m$ such that $v^T A = 0$ with $v \geq 0$ and $v^T b < 0$. The condition $\langle x_i - x_j, y_i - y_j \rangle \geq 0$ expands to the condition $a_{ii} - a_{ij} - a_{ji} + a_{jj} \geq 0$ on the entries of A. Hence,

$$\begin{aligned} 0 &\leq \sum_{i,j} v_i v_j (a_{ii} - a_{ij} - a_{ji} + a_{jj}) \\ &= \sum_j v_j \sum_i v_i a_{ii} - \sum_j v_j \sum_i v_i a_{ij} - \sum_i v_i \sum_j v_j a_{ji} + \sum_i v_i \sum_j v_j a_{jj} \\ &= \sum_j v_j v^T b - \sum_j v_j 0 - \sum_i v_i 0 + \sum_i v_i v^T b = 2 v^T b \sum_i v_i < 0, \end{aligned}$$

which is a contradiction.

Also solved by R. Stong and the proposer.

A Sum of Tails of the Zeta Function

12215 [2020, 853]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+4)^2} + \cdots \right) - \frac{1}{2n} \right).$$

Solution by Gaurav Aggarwal, student, Guru Nanak Dev University, Amritsar, India. The sum equals $\pi^2/16 + 1/2$. Let

$$S_N = \sum_{n=1}^{N} \left(\left(\frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+4)^2} + \cdots \right) - \frac{1}{2n} \right).$$

The term

$$\left(\frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+4)^2} + \cdots\right) - \frac{1}{2n}$$

clearly approaches 0 as n approaches infinity, since the part in parentheses is bounded by $\sum_{k=n}^{\infty} 1/k^2$, which itself goes to 0. Therefore, it suffices to prove

$$\lim_{N\to\infty} S_{2N} = \pi^2/16 + 1/2.$$

We compute

$$\begin{aligned} S_{2N} &= \sum_{i=1}^{N} i\left(\frac{1}{(2i-1)^2} + \frac{1}{(2i)^2}\right) + N\sum_{i=2N+1}^{\infty}\frac{1}{i^2} - \sum_{i=1}^{2N}\frac{1}{2i} \\ &= \sum_{i=1}^{N}\left(\frac{i}{(2i-1)^2} + \frac{i}{(2i)^2} - \frac{1}{2(2i-1)} - \frac{1}{2(2i)}\right) + N\sum_{i=2N+1}^{\infty}\frac{1}{i^2} \\ &= \sum_{i=1}^{N}\frac{1}{2(2i-1)^2} + N\sum_{i=2N+1}^{\infty}\frac{1}{i^2}. \end{aligned}$$

Noting that $\zeta(2) = \pi^2/6$, where ζ is the Riemann zeta function, we have

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{1}{2(2i-1)^2} = \frac{1}{2}\left(1 - \frac{1}{2^2}\right)\zeta(2) = \frac{\pi^2}{16}.$$

We use telescoping series again and the squeeze theorem to show that the remaining term tends to $1/2$:

$$\begin{aligned} \frac{N}{2N+1} &= N\sum_{i=2N+1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right) = N\sum_{i=2N+1}^{\infty}\frac{1}{i(i+1)} < N\sum_{i=2N+1}^{\infty}\frac{1}{i^2} \\ &< N\sum_{i=2N+1}^{\infty}\frac{1}{(i-1)i} = N\sum_{i=2N+1}^{\infty}\left(\frac{1}{i-1} - \frac{1}{i}\right) = \frac{N}{2N} = \frac{1}{2}. \end{aligned}$$

Hence $\lim_{N\to\infty} S_N = \lim_{N\to\infty} S_{2N} = \pi^2/16 + 1/2$.

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (Saudi Arabia), K. N. Boyadzhiev, P. Bracken, B. Bradie, V. Brunetti & A. Aurigemma & G. Bramanti & J. D'Aurizio & D. B. Malesani (Italy), B. S. Burdick, H. Chen, C. Curtis, T. Dickens, G. Fera (Italy), M. L. Glasser, H. Grandmontagne (France), J.-P. Grivaux (France), J. A. Grzesik, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), Y. Jinhai (China), O. Kouba (Syria), K.-W. Lau (China), G. Lavau (France), O. P. Lossers (Netherlands), R. Molinari, A. Natian, M. Omarjee (France), P. Palmieri (Italy), K. Schilling, A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, J. Vukmirović (Serbia), T. Wiandt, Y. Xiang (China), FAU Problem Solving Group, Missouri State Problem Solving Group, and the proposer.

Rotating an Icosahedron

12216 [2020, 944]. *Proposed by Zachary Franco, Houston, TX.* A regular icosahedron with volume 1 is rotated about an axis connecting opposite vertices. What is the volume of the resulting solid?

Solution by Albert Stadler, Herrliberg, Switzerland. It is known (see for example en.wikipedia.org/wiki/Regular_icosahedron) that if the edge length of a regular icosahedron is a, then the radius of the circumscribed sphere is

$$R = \frac{a}{4}\sqrt{10 + 2\sqrt{5}},$$

while the volume is

$$V = \frac{5}{12}(3 + \sqrt{5})a^3.$$

We place the icosahedron in $\mathbb{R}^3$ in such a way that its 12 vertices have the following coordinates:

$$\begin{aligned}
P_1 &: (0, 0, R),\\
P_2\text{–}P_6 &: \frac{R}{\sqrt{5}}\left(2\cos\left(\frac{2k\pi}{5}\right), 2\sin\left(\frac{2k\pi}{5}\right), 1\right), \quad k \in \{0, \dots, 4\},\\
P_7\text{–}P_{11} &: \frac{R}{\sqrt{5}}\left(2\cos\left(\frac{(2k+1)\pi}{5}\right), 2\sin\left(\frac{(2k+1)\pi}{5}\right), -1\right), \quad k \in \{0, \dots, 4\},\\
P_{12} &: (0, 0, -R).
\end{aligned}$$

The segment connecting the two points P_2 and P_7 is given by

$$s(t) = \frac{R}{\sqrt{5}}\left[t\left(2, 0, 1\right) + (1-t)\left(2\cos\left(\frac{\pi}{5}\right), 2\sin\left(\frac{\pi}{5}\right), -1\right)\right], \quad 0 \le t \le 1.$$

This segment generates the boundary of the middle part of the solid formed when the icosahedron is rotated about the z-axis. The other two parts are cones whose boundaries are generated by rotating the segment connecting P_1 and P_2 and the segment connecting P_7 and P_{12}.

The distance of $s(t)$ from the z-axis equals

$$\frac{R}{\sqrt{5}}\left\|t\left(2, 0, 0\right) + (1-t)\left(2\cos\left(\frac{\pi}{5}\right), 2\sin\left(\frac{\pi}{5}\right), 0\right)\right\| = R\sqrt{\frac{4 - 2(3-\sqrt{5})t(1-t)}{5}}.$$

Therefore, the volume of the rotated icosahedron equals

$$V_{\text{rot}} = \frac{2}{3}\pi\left(R - \frac{R}{\sqrt{5}}\right)\left(\frac{2R}{\sqrt{5}}\right)^2 + \pi R^2\frac{2R}{\sqrt{5}}\int_0^1\left(\frac{4 - 2(3-\sqrt{5})t(1-t)}{5}\right)dt.$$

The first term in this formula is the volume of the two cones, and the second is the volume of the middle part. Evaluating the integral and simplifying we obtain

$$V_{\text{rot}} = \frac{2}{15}(5 + \sqrt{5})\pi R^3 = \frac{\sqrt{2}}{240}\left(5 + \sqrt{5}\right)^{5/2}\pi a^3.$$

If the volume of the icosahedron is 1, then a is determined by

$$a^3 = \frac{12}{5(3 + \sqrt{5})}.$$

Substituting this into our formula for V_{rot} gives a volume of

$$V_{\text{rot}} = \frac{\pi}{5}\sqrt{\frac{5 + \sqrt{5}}{2}} \approx 1.19513.$$

Also solved by F. Chamizo (Spain), C. Curtis & J. Boswell, G. Fera (Italy), O. Geupel (Germany), J.-P. Grivaux (France), N. Hodges (UK), M. J. Knight, G. Lavau (France), O. P. Lossers (Netherlands), M. D. Meyerson, R. Stong, D. Terr, T. Wiandt, L. Zhou, Davis Problem Solving Group, Eagle Problem Solvers, and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C6. *Due to R. E. Machol and L. J. Savage, contributed by David Aldous, University of California, Berkeley, CA*. Consider four random points on the surface of a sphere, chosen uniformly and independently. Prove that the probability that the tetrahedron determined by the points contains the center of the sphere is $1/8$.

The Affine Hull of Four Points in Space

C5. *Contributed by the editors*. Given a set S in $\mathbb{R}^n$, let $L(S)$ be the set of all points lying on some line determined by two points in S. For example, if S is the set of vertices of an equilateral triangle in $\mathbb{R}^2$, then $L(S)$ is the union of the three lines that extend the sides of the triangle, and $L(L(S))$ is all of $\mathbb{R}^2$. If S is the set of vertices of a regular tetrahedron, then what is $L(L(S))$?

Solution. There are precisely four points that are not in $L(L(S))$. Inscribe the tetrahedron in a cube with the vertices of the tetrahedron at four of the corners of the cube. The four other corners of the cube are the missing points.

To see that these points are missed, observe that $L(S)$ consists of all the points on the extended edges of the tetrahedron. A line through points on adjacent extended edges lies in the plane of a tetrahedral face and so misses the unused corners. Also, a line connecting one such corner to a nearby extended edge of the tetrahedron lies in the plane of a face of the cube and so misses any of the skew edges.

We now show that all other points in $\mathbb{R}^3$ are included. Let P_1 be the plane containing the top face of the cube and let P_2 be the plane containing the bottom face. Let l_1 and l_2 be the tetrahedral edges lying in P_1 and P_2, respectively. Notice that P_1 is the unique plane containing l_1 that is parallel to l_2, and similarly for P_2. Suppose that Q is a point that does not lie on either P_1 or P_2. Let P be the plane containing Q and l_1. Since Q does not lie on P_1, P is not equal to P_1, so it is not parallel to l_2. Therefore it intersects l_2, say at R. The line QR lies in the plane P, which contains l_1. Since Q does not lie on P_2, QR is not parallel to l_1. Therefore QR must intersect l_1, say at T. But now Q, R, and T are collinear, so Q is in $L(L(S))$.

This argument shows that $L(L(S))$ contains all points that do not lie in either the plane of the top of the cube or the plane of the bottom. Similarly, it contains all points that do not lie on either the plane of the left side or the right side, and all points that do not lie on either the plane of the front or back. This means that the only points that can be missed are the corners of the cube.

Editorial comment. The problem was proposed by Victor Klee as Problem 1413 in *Math. Mag.* 66 (1993) 56, with solution in *Math. Mag.* 67 (1993) 68–69. See also V. Klee (1963), The generation of affine hulls, *Acta Scient. Math. (Szeged)* 24, 60–81.