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Problems and Solutions

Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below must be submitted by December 31, 2022. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12335. *Proposed by Tom Karzes, Sunnyvale, CA, Stephen Lucas, James Madison University, Harrisonburg, VA, and James Propp, University of Massachusetts, Lowell, MA.* A *Gaussian integer* is a complex number z such that $z = a + bi$ for integers a and b. Show that every Gaussian integer can be written in at most one way as a sum of distinct powers of $1 + i$, and that the Gaussian integer z can be expressed as such a sum if and only if $i - z$ cannot.

12336. *Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let N be the center of the nine-point circle of triangle ABC, and let D, E, and F be the orthogonal projections of N onto the sides BC, CA, and AB, respectively. Prove that the Euler lines of triangles ABC, AEF, BFD, and CDE are concurrent. Prove also that the point of concurrency is equidistant from the circumcenters of AEF, BFD, and CDE.

12337. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $k \in \{0, 1, 2\}$, let

$$S_k = \sum \frac{(-4)^n}{2n+1} \binom{2n}{n}^{-1},$$

where the sum is taken over all nonnegative integers n that are congruent to k modulo 3. Prove

(a) $S_0 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} + \dfrac{\pi}{6}$;

(b) $S_1 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} - \dfrac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - \dfrac{\pi}{12}$; and

(c) $S_2 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} + \dfrac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - \dfrac{\pi}{12}$.

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12338. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^\infty \frac{\cos(x)-1}{x\ (e^x-1)}\,dx = \frac{1}{2}\ln\bigl(\pi\ \mathrm{csch}(\pi)\bigr).$$

12339. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Let A and B be complex n-by-n matrices for which $A^2 + xB^2 = y\,(AB - BA)$, where x is a positive real number and y is a real number such that $(1/\pi)\cos^{-1}\bigl((y^2-x)/(y^2+x)\bigr)$ is irrational. Prove that $(AB - BA)^n$ is the zero matrix.

12340. *Proposed by Antonio Garcia, Strasbourg, France.* Let $g : [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n}\int_0^1 \frac{g(x)}{x^n + (1-x)^n}dx = Cg(1/2)$$

for some constant C (independent of g) and determine the value of C.

12341. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let $x_1, \dots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i^2 \le n$, and let $S = \sum_{i=1}^n x_i$. Prove

$$\prod_{i=1}^n \left(1 + \frac{1}{x_i x_{i+1}}\right)^{x_i^2} \ge 2^{S^2/n},$$

where x_{n+1} is taken to be x_1.

SOLUTIONS

A Common Coefficient

12209 [2020, 852]. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.* Prove

$$\sum_{k=0}^n (-1)^k \binom{n}{k}\binom{m+2n-2k+1}{m} = \sum_{k=0}^n \binom{n}{k}\binom{m+k+1}{m-n}$$

for all integers m and n with $m \ge n \ge 0$.

Solution by Michel Bataille, France. We show that both sides equal the coefficient of x^m in the polynomial P defined by

$$P(x) = (1+x)^{m+1}(2x+x^2)^n = (1+x)^{m+1}((1+x)^2-1)^n.$$

Using the binomial theorem twice yields

$$P(x) = (1+x)^{m+1}\sum_{k=0}^n (-1)^k\binom{n}{k}(1+x)^{2(n-k)} = \sum_{k=0}^n (-1)^k\binom{n}{k}(1+x)^{2n-2k+m+1}$$

$$= \sum_{k=0}^n (-1)^k\binom{n}{k}\sum_{j=0}^{2n-2k+m+1}\binom{2n-2k+m+1}{j}x^j.$$

This expresses the left side of the identity as the coefficient of x^m in the expansion of $P(x)$.

Also,

$$P(x) = (1+x)^{m+1}(x(2+x))^n = x^n(1+x)^{m+1}(1+(1+x))^n,$$

so another two uses of the binomial theorem yield

$$P(x) = x^n(1+x)^{m+1}\sum_{k=0}^n\binom{n}{k}(1+x)^k = \sum_{k=0}^n\binom{n}{k}\sum_{j=0}^{m+k+1}\binom{m+k+1}{j}x^{n+j}.$$

This shows that the coefficient of x^m in the expansion of $P(x)$ is also the right side of the identity, completing the proof.

Also solved by R. Boukharfane (Saudi Arabia), Ó. Ciaurri (Spain), J. Boswell & C. Curtis, G. Fera (Italy), N. Hodges (UK), M. Kaplan & M. Goldenberg, O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), M. Maltenfort, E. Schmeichel, A. Stadler (Switzerland), R. Stong, F. A. Velandia (Colombia), M. Vowe (Switzerland), J. Vukmirović (Serbia), J. Wangshinghin, M. Wildon (UK), X. Ye (China), and the proposer.

A Median Inequality

12214 [2020, 853]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let x, y, and z be the lengths of the medians of a triangle with area F. Prove

$$\frac{xyz(x+y+z)}{xy+zx+yz} \geq \sqrt{3}F.$$

Solution by Oliver Geupel, Brühl, Germany. The Cauchy–Schwarz inequality implies that $x^2+y^2+z^2 \geq xy+yz+zx$, and therefore

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx). \tag{1}$$

It is well known that the medians of a triangle with area F are the sides of a triangle with area $K = 3F/4$ (see, for example, sections 91–93 in N. Altschiller-Court (1952), *College Geometry*, New York: Barnes and Noble). Moreover, it is known that a triangle with sides x, y, and z and area K satisfies the inequality

$$\frac{9xyz}{x+y+z} \geq 4\sqrt{3}K \tag{2}$$

(see item 4.13 on p. 45 of O. Bottema et al. (1969), *Geometric Inequalities*, Groningen: Wolters-Noordhoff). Combining (1) and (2), we obtain

$$\frac{xyz(x+y+z)}{xy+yz+zx} \geq \frac{3xyz(x+y+z)}{(x+y+z)^2} = \frac{3xyz}{x+y+z} \geq \frac{4\sqrt{3}K}{3} = \sqrt{3}F.$$

Editorial comment. Inequality (2) appeared as part of elementary problem E1861 [1966, 199; 1967, 724] from this MONTHLY, proposed by T. R. Curry and solved by Leon Bankoff. The equation $K = 3F/4$ is also featured as Theorem 10.4 on p. 165 of C. Alsina and R. B. Nelsen (2010), *Charming Proofs: A Journey Into Elegant Mathematics*, Washington, DC: Mathematical Association of America.

Also solved by A. Alt, H. Bai (Canada), M. Bataille (France), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), I. Borosh, R. Boukharfane (Saudi Arabia), P. Bracken, S. H. Brown, C. Curtis, N. S. Dasireddy (India), A. Dixit (India) & S. Pathak (UK), H. Y. Far, G. Fera (Italy), N. Hodges (UK), W. Janous (Austria), M. Kaplan & M. Goldenberg, P. Khalili, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), M. Lukarevski (Macedonia), A. Pathak (India), C. R. Pranesachar (India), C. Schacht, V. Schindler (Germany), A. Stadler (Switzerland), N. Stanciu & M. Drăgan (Romania), R. Stong, B. Suceavă, M. Vowe (Switzerland), J. Vukmiroviıc (Serbia), T. Wiandt, X. Ye (China), M. R. Yegan (Iran), Davis Problem Solving Group, and the proposer.

Another Incenter-Centroid Inequality

12217 [2020, 944]. *Proposed by Giuseppe Fera, Vicenza, Italy.* Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$\frac{3}{2} < \frac{AI}{AG} + \frac{BI}{BG} + \frac{CI}{CG} \leq 3.$$

Solution by Haoran Chen, Suzhou, China. Let $a = BC$, $b = CA$, and $c = AB$. Also let $s = (a + b + c)/2$. Let m_a be the length of the median from A, r the radius of the incircle, and K the point of tangency of the incircle with AB. By the triangle inequality,

$$2m_a < \left(\frac{a}{2} + b\right) + \left(\frac{a}{2} + c\right) = 2s.$$

Also, $AG = 2m_a/3$ and $AI > AK = s - a$. Therefore

$$\frac{AI}{AG} = \frac{3AI}{2m_a} > \frac{3(s-a)}{2s}.$$

Summing this with the other two analogous inequalities establishes the strict lower bound of $3/2$.

For the upper bound, note that

$$rs = \text{area of } \triangle ABC = \frac{bc \sin A}{2},$$

and therefore

$$AI^2 = \frac{AK}{\cos(A/2)} \cdot \frac{r}{\sin(A/2)} = \frac{(s-a)r}{(1/2)\sin A} = \frac{bc(s-a)}{s}.$$

Also, by Apollonius's theorem,

$$4m_a^2 = 2b^2 + 2c^2 - a^2 = (b + c + a)(b + c - a) + (b - c)^2 \geq 4s(s - a).$$

Therefore

$$\frac{AI}{AG} = \frac{3AI}{2m_a} \leq \frac{3\sqrt{bc}}{2s} \leq \frac{3(b+c)}{4s}.$$

Summing this with the other two analogous inequalities establishes the upper bound of 3.

Editorial comment. Problem 12175 [2020, 372; 2021, 952] establishes

$$\frac{AI^2}{AG^2} + \frac{BI^2}{BG^2} + \frac{CI^2}{CG^2} \leq 3.$$

This can be used to give an alternative proof of the upper bound: By the Cauchy–Schwarz inequality,

$$\frac{AI}{AG} + \frac{BI}{BG} + \frac{CI}{CG} \leq \sqrt{3\left(\frac{AI^2}{AG^2} + \frac{BI^2}{BG^2} + \frac{CI^2}{CG^2}\right)} \leq 3.$$

Also solved by A. Alt, S. Gayen (India), P. Khalili, S. Lee (Korea), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, and the proposer.

Composing All Permutations of $[n]$ to Do Nothing

12218 [2020, 944]. *Proposed by Richard Stong, Center for Communications Research, La Jolla, CA, and Stan Wagon, Macalester College, St. Paul, MN.* For which positive integers n does there exist an ordering of all permutations of $\{1, \ldots, n\}$ so that their composition in that order is the identity?

Solution by S. M. Gagola Jr., Kent State University, Kent, OH. Such an ordering of permutations is possible for $n = 1$ (trivially) and for all n at least 4.

When n is 2 or 3, the number of permutations with odd parity is odd, so no composition in these cases can have even parity like the identity. Note, however, that when $n = 3$ the product of the three distinct transpositions always equals the middle factor ($t_1 t_2 t_3 = t_2$).

Before considering $n \geq 4$, it is useful to note that any group of even order has an odd number of elements of order 2. To see this, pair the elements of the group with their inverses. The identity element and the elements of order two (involutions) are self-paired, while the remaining elements form sets of size 2. Since the group has even order, the number of involutions is therefore odd.

If in a group of even order a product of the involutions (in some order) can be shown to equal the identity, then the remaining elements can be paired with their inverses to yield a product of all the elements equaling the identity. Hence it suffices to show that for $n \geq 4$, the involutions of the symmetric group S_n can be ordered so that their product is the identity.

The nine involutions in S_4 can be partitioned into three triples as follows:

$$\{(12), (34), (12)(34)\}, \quad \{(13), (24), (13)(24)\}, \quad \{(14), (23), (14)(23)\}.$$

The product of the three involutions in any one subset (in any order) equals the identity; this completes the $n = 4$ case.

For $n = 5$, we partition the involutions in S_5 into sets $I_1, \dots, I_5$ and order each set to obtain a product yielding the identity. For I_1 we take the nine involutions on $\{2, 3, 4, 5\}$. By the $n = 4$ case, there is a product of these yielding the identity. For $j \geq 2$, let I_j consist of all involutions that exchange 1 and j. One element is $(1j)$, and each of the other three elements is the product of $(1j)$ and a transposition of two of the three elements of $\{2, 3, 4, 5\} - \{j\}$. Each of the four elements of I_j transposes 1 and j, and we have noted that the product of the three transpositions on a set of size 3 can be ordered to yield any one of the three transpositions. We can therefore choose orderings of each of I_2, I_3, I_4, and I_5 so that their products are (45), (45), (23), and (23), respectively. Combining these orderings completes the $n = 5$ case.

The solutions for $n = 4$ and $n = 5$ provide a basis for a proof by induction. We write $[n]$ for $\{1, \dots, n\}$. For $n \geq 6$, partition the involutions of S_n into the n sets $I_1, \dots, I_n$, where I_1 consists of all the involutions on $[n] - \{1\}$, and I_j for $j \geq 2$ consists of all involutions exchanging 1 and j. The $n - 1$ case yields an ordering of I_1 that produces the identity. For $j \geq 2$, each element of I_j consists of the transposition $(1j)$ times an element of the symmetric group on $[n] - \{1, j\}$ that is the identity or an involution. As noted earlier, I_j thus has even size, and hence any product of the elements of I_j leaves 1 and j in place. Furthermore, the $n - 2$ case guarantees that the elements of I_j other than $(1j)$ can be ordered so that their effect on $[n] - \{1, j\}$ is the identity. Doing this independently for all I_j completes the proof.

Editorial comment. The problem is a special case of a result from J. Dénes and P. Hermann (1982), On the product of all elements in a finite group, in E. Mendelsohn, ed., *Algebraic and geometric combinatorics*, North-Holland Math. Stud. 65, Amsterdam: North-Holland, pp. 105–109. A special case of their theorem that still includes the problem here is proved more simply in M. Vaughan-Lee and I. M. Wanless (2003), Latin squares and the Hall–Paige conjecture. *Bull. London Math. Soc.* 35, no. 2, 191–195.

The solver Gagola noted that if a group G of even order has a cyclic Sylow 2-subgroup, then there is a normal 2-complement N, and the product of the elements of G taken in any order always represents a coset of order 2 in the factor group G/N. Therefore, this product can never equal the identity element. He then asked whether a group of even order that does not have a cyclic Sylow 2-subgroup always has an ordering of the elements so that the resulting product produces the identity. As Vaughan-Lee and Wanless wrote, "The Hall–Paige conjecture deals with conditions under which a finite group G will possess a complete mapping, or equivalently a Latin square based on the Cayley table of G will possess a transversal. Two necessary conditions are known to be: (i) that the Sylow 2-subgroups of G are trivial or noncyclic, and (ii) that there is some ordering of the elements of G which yields a trivial product. These two conditions are known to be equivalent, but the first direct, elementary proof that (i) implies (ii) is given here." Thus the answer to Gagola's question is yes.

Also solved by F. Chamizo & Y. Fuertes (Spain), D. Dima (Romania), O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin (USA) & B. M. Bekker (Russia), O. P. Lossers (Netherlands), M. Reid, A. Stadler (Switzerland), R. Tauraso (Italy), T. Wilde (UK), and the proposers.

A Vanishing Sum of Stirling Numbers

12219 [2020, 944]. *Proposed by Brad Isaacson, New York City College of Technology, New York, NY.* Let k and m be positive integers with $k < m$. Let $c(m,k)$ be the number of permutations of $\{1,\dots,m\}$ consisting of k cycles. (The numbers $c(m,k)$ are known as unsigned Stirling numbers of the first kind.) Prove

$$\sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} c(j,k)}{(j-1)!} = 0$$

whenever m and k have opposite parity.

Solution by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy. Let

$$F_m(x) = \sum_{k=1}^{m} (-x)^k \sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} c(j,k)}{(j-1)!}.$$

Here $F_m(x)$ is a generating function for the desired sum, evaluated at the negative of the formal variable. We aim to show that the coefficients of odd powers of x are 0 when m is even, and the coefficients of even powers of x are 0 when m is odd. For this it suffices to show

$$F_m(-x) = (-1)^m F_m(x).$$

The well-known generating function for the unsigned Stirling numbers of the first kind is given by $\sum_{k=1}^{j} c(j,k) y^k = \prod_{i=0}^{j-1}(y+i)$ (easily proved combinatorially). Setting $y = -x$ yields $\sum_{k=1}^{j} (-1)^{j-k} c(j,k) x^k = \prod_{i=0}^{j-1}(x-i)$.

We interchange the order of summation to take advantage of this identity. Let x be an integer with $x \ge m$. We compute

$$\begin{aligned} F_m(x) &= \sum_{j=1}^{m} \frac{2^j \binom{m}{j}}{(j-1)!} \sum_{k=1}^{j} (-1)^{j-k} c(j,k) x^k = \sum_{j=1}^{m} \frac{2^j \binom{m}{j}}{(j-1)!} \prod_{i=0}^{j-1}(x-i) \\ &= m \sum_{j=1}^{m} 2^j \binom{m-1}{j-1}\binom{x}{j} = m \sum_{j=1}^{m} \binom{m-1}{m-j}\binom{x}{j} 2^j \\ &= m[z^m](1+z)^{m-1}(1+2z)^x = m[z^m](1+z)^{x+m-1}\left(1+\frac{z}{1+z}\right)^x, \end{aligned}$$

where $[z^m]$ is the "coefficient operator" extracting the coefficient of z^m in the expression that follows it.

To extract the coefficient of z^m in a different way, we apply the binomial theorem twice to obtain

$$(1+z)^{x+m-1}\left(1+\frac{z}{1+z}\right)^x = \sum_{j=0}^{x}(1+z)^{x+m-j-1}\binom{x}{j}z^j$$
$$= \sum_{j=0}^{x}\binom{x}{j}z^j \sum_{k=0}^{x+m-j-1}\binom{x+m-j-1}{k}z^k.$$

To extract all the contributions to the coefficient of z^m, restrict j to run from 0 to m, and set $k = m - j$ in the inner sum. This leads to the formula

$$F_m(x) = m[z^m](1+z)^{x+m-1}\left(1+\frac{z}{1+z}\right)^x = m\sum_{j=0}^{m}\binom{x+m-j-1}{m-j}\binom{x}{j}.$$

Viewing $\binom{x}{j}$ as a polynomial in x, this is a polynomial equation that holds for every integer x with $x \geq m$. It therefore holds for all real numbers x. Thus, by reversing the index of summation and using

$$\binom{-y}{r} = (-1)^r\binom{y+r-1}{r},$$

we obtain

$$F_m(-x) = m\sum_{j=0}^{m}\binom{-x+m-j-1}{m-j}\binom{-x}{j} = m\sum_{j=0}^{m}\binom{-(x-j+1)}{j}\binom{-x}{m-j}$$
$$= m\sum_{j=0}^{m}(-1)^j\binom{x}{j}\cdot(-1)^{m-j}\binom{x+m-j-1}{m-j} = (-1)^m F_m(x),$$

as desired.

Editorial comment. In addition to the polynomials studied above, solvers used induction, contour integration, generating function manipulations, or primitive Dirichlet characters.

There is a direct combinatorial proof of the needed identity

$$\sum_{j=1}^{m}2^j\binom{m-1}{j-1}\binom{x}{j} = \sum_{j=0}^{m}\binom{x+m-j-1}{m-j}\binom{x}{j}$$

in the proof given above. Both sides count the distinguishable ways to place m balls in x boxes, where balls may be black or white, with each box having at most one white ball but any number of black balls. On the left side, j is the number of boxes that have balls: Pick the boxes, distribute the balls with a positive number in each chosen box, and decide for each chosen box whether to make one of the balls white. On the right side, j is the number of white balls: Pick boxes for them, and independently distribute $m - j$ black balls into the x boxes with repetition allowed.

Also solved by N. Hodges (UK), O. Kouba (Syria), P. Lalonde (Canada), A. Stadler (Switzerland), J. Wangshinghin (Canada), and the proposer.

A Limit Related to the Basel Problem

12220 [2020, 944]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $a_n = \sum_{k=1}^{n} 1/k^2$ and $b_n = \sum_{k=1}^{n} 1/(2k-1)^2$. Prove

$$\lim_{n\to\infty} n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{3}{\pi^2}.$$

Solution by Charles Curtis, Missouri Southern State University, Joplin, MO. Note that

$$b_n = \sum_{k=1}^{2n} \frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{n} \frac{1}{k^2} = \frac{3}{4}\sum_{k=1}^{n} \frac{1}{k^2} + \sum_{k=n+1}^{2n} \frac{1}{k^2} = \frac{3}{4}a_n + \sum_{k=n+1}^{2n} \frac{1}{k^2}.$$

Therefore

$$n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{n}{a_n}\sum_{k=n+1}^{2n} \frac{1}{k^2} = \frac{n}{a_n}\sum_{k=1}^{n} \frac{1}{(n+k)^2} = \frac{1}{a_n}\left[\frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+k/n)^2}\right].$$

It is well known that a_n converges to $\pi^2/6$ (this is often called the *Basel problem*). The expression in square brackets can be interpreted as a Riemann sum, yielding

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+k/n)^2} = \int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}.$$

Hence we get the desired result.

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), F. R. Ataev (Uzbekistan), M. Bataille (France), N. Batir (Turkey), A. Berkane (Algeria), N. Bhandari (Nepal), R. Boukharfane (Morocco), P. Bracken, B. Bradie, V. Brunetti & J. Garofali & A. Aurigemma (Italy), F. Chamizo (Spain), H. Chen, C. Chiser (Romania), G. Fera (Italy), D. Fleischman, O. Geupel (Germany), D. Goyal (India), N. Grivaux (France), J. A. Grzesik, L. Han, J.-L. Henry (France), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), R. Howard, W. Janous (Austria), O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), G. Lavau (France), S. Lee, P. W. Lindstrom, O. P. Lossers (Netherlands), C. J. Lungstrom, J. Magliano, R. Molinari, A. Natian, S. Omar (Morocco), M. Omarjee (France), M. Reid, S. Sharma (India), J. Singh (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, M. Tang, R. Tauraso (Italy), D. Terr, D. B. Tyler, D. Văcaru (Romania), J. Vinuesa (Spain), M. Vowe (Switzerland), J. Wangshinghin (Canada), T. Wiandt, Q. Zhang (China), Missouri State University Problem Solving Group, and the proposer.

A Logarithmic Integral Evaluated by Residues

12221 [2020, 945]. *Proposed by Necdet Batır, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey.* Prove

$$\int_0^1 \frac{\log(x^6+1)}{x^2+1}\,dx = \frac{\pi}{2}\log 6 - 3G,$$

where G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. Let I denote the requested integral. Writing I as a sum of two integrals and then making the change of variable $t = 1/x$ in the first integral, we obtain

$$I = \int_0^1 \frac{\log(1+1/x^6)}{1+x^2}\,dx + 6\int_0^1 \frac{\log x}{1+x^2}\,dx = \int_1^\infty \frac{\log(1+t^6)}{1+t^2}\,dt + 6\int_0^1 \frac{\log x}{1+x^2}\,dx,$$

and therefore

$$2I = \int_0^\infty \frac{\log(1+x^6)}{1+x^2}\,dx + 6\int_0^1 \frac{\log x}{1+x^2}\,dx.$$

To evaluate the last integral, we express $1/(1+x^2)$ as an infinite series:

$$\int_0^1 \frac{\log x}{1+x^2}\,dx = \int_0^1 \left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right)\log x\,dx.$$

Since the partial sums of the series are bounded in absolute value by 1, the dominated convergence theorem justifies interchanging the order of summation and integration, and then an integration by parts yields

$$\int_0^1 \frac{\log x}{1+x^2}\,dx = \sum_{k=0}^{\infty}(-1)^k \int_0^1 x^{2k}\log x\,dx = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)^2} = -G.$$

Thus,

$$2I = \int_0^\infty \frac{\log(1+x^6)}{1+x^2}\,dx - 6G,$$

so the required result follows from

$$\int_{-\infty}^{\infty} \frac{\log(1+x^6)}{1+x^2}\,dx = 2\pi\log 6, \tag{1}$$

which we now prove using the method of residues.

For $z = |z|e^{i\theta}$ with $|z| > 0$ and $-\pi < \theta \le \pi$, define $\operatorname{Log} z = \log|z| + i\theta$. The function $\operatorname{Log} z$ is analytic on the open upper half-plane. For $R > 1$ let C_R denote the contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Let

$$P_1(z) = z + i, \quad P_2(z) = z - \sqrt{3}/2 + i/2, \quad \text{and} \quad P_3(z) = z + \sqrt{3}/2 + i/2.$$

For $j \in \{1, 2, 3\}$, the function $\operatorname{Log} P_j(z)$ is analytic on the closed upper half-plane, and therefore the residue theorem yields

$$\begin{aligned}\int_{-R}^{R} \frac{\operatorname{Log} P_j(x)}{1+x^2}\,dx + \int_{C_R} \frac{\operatorname{Log} P_j(z)}{1+z^2}\,dz &= 2\pi i \operatorname{Res}\left(\frac{\operatorname{Log} P_j(z)}{1+z^2}, i\right)\\ &= \pi \operatorname{Log} P_j(i).\end{aligned} \tag{2}$$

Since

$$\left|\int_{C_R} \frac{\operatorname{Log} P_j(z)}{1+z^2}\,dz\right| \le \pi R\frac{(\log(R+1)+\pi)}{R^2-1},$$

letting $R \to \infty$ in (2) and then taking the real part of the resulting identity yields

$$\int_{-\infty}^{\infty} \frac{\log|P_j(x)|}{1+x^2}\,dx = \pi\log|P_j(i)|.$$

Finally, since

$$\begin{aligned}x^6+1 &= \left(x^2+1\right)\left(x^2-\sqrt{3}x+1\right)\left(x^2+\sqrt{3}x+1\right)\\ &= \left(x^2+1\right)\left((x-\sqrt{3}/2)^2+1/4\right)\left((x+\sqrt{3}/2)^2+1/4\right)\\ &= |P_1(x)|^2|P_2(x)|^2|P_3(x)|^2,\end{aligned}$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\log(1+x^6)}{1+x^2}\,dx &= \sum_{j=1}^{3}\int_{-\infty}^{\infty}\frac{2\log|P_j(x)|}{1+x^2}\,dx\\ &= \sum_{j=1}^{3} 2\pi\log|P_j(i)| = 2\pi\left(\log 2 + \log\sqrt{3} + \log\sqrt{3}\right)\\ &= 2\pi\log 6,\end{aligned}$$

which completes the proof of (1).

Editorial comment. Several solvers noted that a similar problem appeared as problem 2107 in *Math. Mag.* 93 (2020), p. 389.

Also solved by U. Abel & V. Kushnirevych (Germany), F. R. Ataev (Uzbekistan), M. Bataille (France), A. Berkane (Algeria), N. Bhandari (Nepal), K. N. Boyadzhiev, P. Bracken, B. Bradie, V. Brunetti & J. Garofali & J. D'Aurizio (Italy), H. Chen, B. E. Davis, G. Fera (Italy), M. L. Glasser, R. Gordon, H. Grandmontagne (France), J. A. Grzesik, L. Han, D. Henderson, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), P. Khalili, O. Kouba (Syria), Z. Lin (China), O. P. Lossers (Netherlands), T. M. Mazzoli (Austria), M. Omarjee (France), V. Schindler (Germany), J. Singh (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), D. Văcaru (Romania), T. Wiandt, M. R. Yegan (Iran), and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C7. *Contributed by Alan D. Taylor, Union College, Schenectady, NY.* Are the additive group of real numbers and the additive group of complex numbers isomorphic?

Random Tetrahedra Inscribed in a Sphere

C6. *Contributed by David Aldous, University of California, Berkeley, CA.* Consider four random points on the surface of a sphere, chosen uniformly and independently. Prove that the probability that the tetrahedron determined by the points contains the center of the sphere is $1/8$.

Solution. Assume the sphere is in $\mathbb{R}^3$ centered at the origin O. Fix the point P_4 and then choose P_1, P_2, P_3 by randomly choosing three diameters, D_1, D_2, and D_3, and then choosing, randomly, an end of each. There are eight ways to choose the endpoints. The probability conclusion follows from the observation that, for almost all choices of diameters, exactly one of the eight choices of endpoints yields a tetrahedron containing O.

To see this, assume that P_1, P_2, and P_3 are chosen so that no three of the points P_1, P_2, P_3, P_4 are linearly dependent as vectors in $\mathbb{R}^3$. (The opposite case has probability 0.) The equation $-P_4 = xP_1 + yP_2 + zP_3$ has a unique solution in nonzero real numbers x, y, and z. Write this as $O = xP_1 + yP_2 + zP_3 + P_4$. The eight choices of endpoints now correspond to the eight choices of signs in the expression $O = \pm xP_1 \pm yP_2 \pm zP_3 + P_4$. The tetrahedron contains O if and only if there is a representation $O = a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4$ where $a_i > 0$ for all i. This happens if and only if the coefficients $\pm x$, $\pm y$, $\pm z$ are all positive, and that occurs for exactly one of the eight equally likely choices.

Editorial comment. This was problem A6 on the 1992 Putnam Competition. For a geometric explanation of what is happening, see the 3blue1brown video "The hardest problem on the hardest test" at youtube.com/watch?v=OkmNXy7er84. In J. G. Wendel (1962), A problem in geometric probability, *Math. Scand.* 11: 109–111, it is proved that for k points on the sphere in $\mathbb{R}^n$, the probability $p_{n,k}$ that the convex hull of the points contains the origin is $\sum_{j=n}^{k-1}\binom{k-1}{j}\big/2^{k-1}$. A corollary is the surprising duality formula $p_{m,m+n} + p_{n,m+n} = 1$. According to Wendel, the problem goes back to R. E. Machol and was first solved by L. J. Savage.

Some further generalizations can be found in R. Howard and P. Sisson (1996), Capturing the origin with random points: Generalizations of a Putnam problem, *College Math. J.*, 27(3): 186–192.