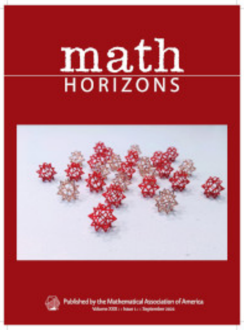


Math Horizons

ISSN: (Print) (Online) Journal homepage: https://maa.tandfonline.com/loi/umho20

The Playground

To cite this article: (2022) The Playground, Math Horizons, 30:1, 30-33, DOI: 10.1080/10724117.2022.2092341

To link to this article: https://doi.org/10.1080/10724117.2022.2092341

Published online: 25 Aug 2022.

THE PLAYGROUND

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Limitations of a Limit Theorem (P438). George Stoica (Saint John, Canada) proposed this problem. Let a_i be a sequence of real numbers. A standard calculus theorem states that if $\lim_{n\to\infty}\sum_{i=1}^{n} a_i$ converges then $\lim_{i\to\infty} a_i = 0$. Prove that the conclusion $\lim_{i\to\infty} a_i = 0$ can be false under the weaker hypothesis that $\lim_{n\to\infty}\sum_{i=\lfloor n/2\rfloor+1}^{n} a_i$ exists.

THE MONKEY BARS

These open-ended problems don't have a previously-known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Consecutively and Squarely Correct (P439). George Berzsenyi (Rose-Hulman Institute of Technology and the first editor of this column) suggests the next problem, which is attributed to Bernard Recamán (Bogotá, Colombia). A positive integer is *squarely correct* if it is a perfect square or if its base-10 representation consists entirely of adjacent blocks of digits that are positive perfect squares. For example, 99 and 100 are two consecutive numbers that are both squarely correct. However, 101 is not squarely correct—all-zero blocks are not allowed.

1) Are there infinitely many pairs of consecutive squarely correct numbers?
2) Is it possible to find three or more consecutive squarely correct numbers?

THE ZIP LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip Line problems require you to read the corresponding article, but doing so can never hurt, of course.

***e*-rational Multiplication (P440).** This problem from Christopher Havens (Twin Rivers PMP) connects to his article with Amy Shell-Gellasch "Fibonacci Meets the Pharaohs: The Decomposition of Multiplication" (p. 24). Christopher asks us to consider a multiplication based on Euler's constant e.

Let s be a real number and

$$s = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

be its continued fraction expansion where the nth convergent of s is given by $p_n / q_n = [a_0; a_1, a_2, \ldots, a_n]$. The *s-Ostrowski decomposition* of a natural number m consists of an integer t and a unique sequence $(c_k)_{k=0}^{t}$ with

$$m = \sum_{k=0}^{t} c_k q_k = c_0 q_0 + c_1 q_1 + \cdots + c_t q_t$$

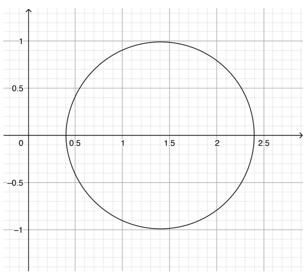
such that $0 \le c_k \le a_{k+1}$ for all k, $0 \le c_0 < a_1$, and for all $k > 0$, if $c_k = a_{k+1}$ then $c_{k-1} = 0$.

Let $s = e = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$. Find the e-Ostrowski decomposition of 179 and use it to compute the product of 179 and 42 following the suggestion from the article that Fibonacci multiplication can be adapted to other recursively defined sequences.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Figure 1. A circular graph?



Circle Sleuth (P441). Gregory Dresden (Washington & Lee University) plotted the graph of the polar curve $r = \sqrt{\cos 2\theta} + (7\,/\,5)\cos\theta$ in the Cartesian plane for values of θ where $\sqrt{\cos 2\theta}$ exists, shown in figure 1. Determine (with justification) whether the graph is a circle.

THE CAROUSEL—OLDIES, BUT GOODIES

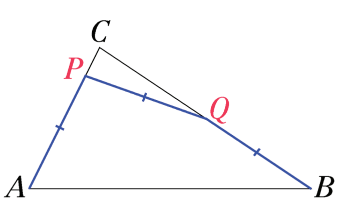
In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Integer Swap Polynomials (C38). Suppose that $g(x)$ is a polynomial with integer coefficients that swaps distinct integers a and b: $g(a) = b$ and $g(b) = a$.

1) Prove that if $g(t) = t$ for some integer t, then $a + b$ is even and $t = (a + b)\,/\,2$.
2) Show that for any distinct integers a and b with $a + b$ even, there exists such a function $g(x)$ both with and without a fixed point.

FEBRUARY WRAP-UP

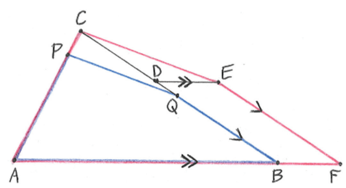
Figure 2. An example of the construction called for in Uniform Three-step.



Uniform Three-step (P422). António Guedes de Oliveira (University of Porto, Portugal) posed this classical ruler-and-compass construction challenge. Given an arbitrary triangle ABC, construct points P and Q on rays $\overrightarrow{AC}$ and $\overrightarrow{BC}$, respectively, so that all three segments AP, PQ, and QB are congruent (as in the example in figure 2).

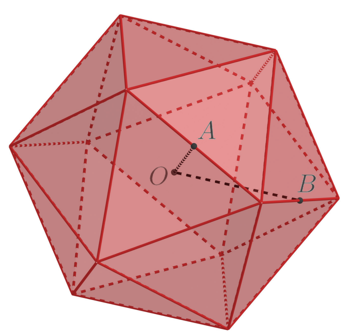
We received a solution from Angelo Rosso and Mihaela Dobrescu of Christopher Newport University and one from a PMP member in Texas. The approaches of these and of the proposer's solution are all completely different. The CNU team performed a trigonometric calculation with ruler and compass and the proposer found P by intersecting AC with the Apollonian circle through B with respect to A and a point O, which is the center of rotation taking ray AC to ray BC. We present the elegant solution from Texas.

Figure 3. Homothetic expansion about A so that the first uniform step is AC (drawn by problem solver).



Imagine scaling the given triangle via a homothety through A (that is, keeping A fixed and scaling along all rays from A) such that AC is exactly the first of the three uniform steps, as illustrated in figure 3. This scaling moves B to F and Q to E such that $AC = CE = EF$. But note that if D is the point on ray BC such that $AC = BD$, then $BFED$ is a parallelogram. Hence, mark off this D, construct the parallel to AB through D, and intersect that parallel with the circle about C through A to find point E. The desired point Q is then the intersection of BC and AE (and P can be found by marking length BQ off on ray AC).

Figure 4. An icosahedron with the central angle formed by the midpoints of two edges.



Icocentral Angle (P430). Bjorn Poonen (MIT) proposed this problem. Let point O be the center of regular icosahedron I. Let A and B be the midpoints of any two edges of I, as in the example in figure 4. Show that the measure of angle AOB in degrees is an integer. What integers can it be?

We received solutions from Katherine Nogin (Clovis North HS), Randy K. Schwartz (Schoolcraft College), problem-solving groups at Christopher Newport University and Georgia Southern

University, and from the PMP member in Texas. The latter provided the proof without words (or very nearly so) in figure 5 that shows the possible angles are all the multiples of 36°, 60°, and 90°.

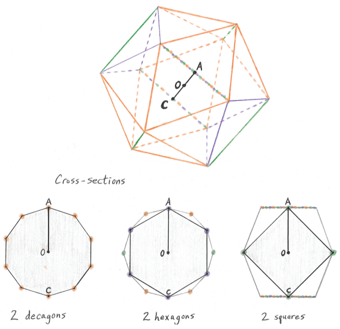
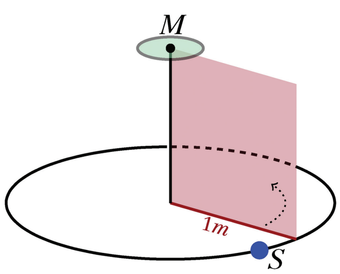


Figure 5. A regular icosahedron with its three possible cross sections containing noncollinear edge midpoints.

Möbius Malachite (P431). This problem came to us courtesy of Randy K. Schwartz (Schoolcraft College). The Möbius Malachite is displayed on a platform at the center of a circular chamber of radius one meter. The platform is supported by a pole (of negligible diameter) containing a laser detection beam that sweeps around the room at a constant rate of once per second (see figure 6).

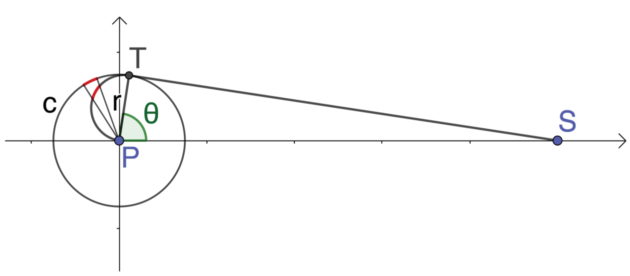
Figure 6. Robot scooter *S* preparing to reach the pole supporting Möbius Malachite *M*.



You want to construct a robot scooter that can enter the chamber and reach the base of the pole without ever intersecting the beam. Assuming the scooter travels at a constant speed along a path of your choice, what's the *slowest* it can move and still reach the pole?

The proposer and the PMP member from Texas suggest the following identical speed and essentially identical path as optimal, offering intuitive justifications. A proof of optimality (or a better path) would be welcome.

Figure 7. A proposed scooter path.



The scooter's strategy is, immediately as the laser passes point S, to embark on a tangent ST to the circle c around the pole P of radius $r = s/\tau$, where s is the scooter's speed. (That circle is chosen because along c the scooter is just fast enough to keep ahead of the rotating laser.) Then the speed s is chosen so that the scooter arrives at T just as the laser has made one full revolution (τ radians) plus $\theta = \angle SPT$ (see figure 7).

This strategy leads to the equation

$$r(\tau+\theta) = s(1+\theta/\tau) = ST = r\tan\theta,$$

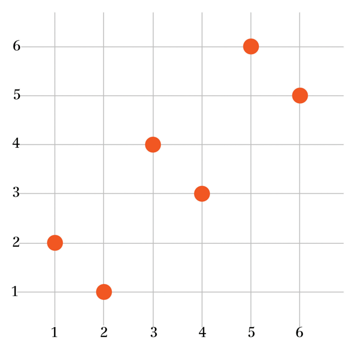
or simply $\tan\theta = \tau + \theta$. Both solvers then determine θ numerically (approximately 1.442 radians) leading to $s \approx 0.8066$ m/s—well below the 1 m/s required to dash directly from S to P.

Incidentally, how does the scooter then make it to the pole rather than endlessly orbit at a distance r from the pole? As the proposer points out, figure 7 also shows (in red) that the scooter may follow the semicircle with diameter TP because in any angular increment around P, the arclength along c is equal to the arclength along the semicircle.

Permutation Rotation (P432). Lara Pudwell posed these questions related to her article "The Hidden and Surprising Structure of Ordered Lists." One can think of a permutation p on n objects as a function from $\{1,\ldots,n\}$ to itself that takes on every value in $\{1,\ldots,n\}$. (In other words, p is onto; and as n is finite, p must be a bijection.) The *graph* of a permutation is the collection of points $\{(i,p(i)) : 1 \le i \le n\}$ in the xy-plane.

1) How many permutations on six objects have a graph that is unchanged after rotation by 180 degrees (about the point (7/2, 7/2), specifically)? Figure 8 shows an example permutation with this property.
2) In general, find (with proof) the number of permutations on n objects that have this property (with respect to the appropriate center of rotation).

Figure 8. The graph of permutation 214365, which has twofold rotational symmetry.



Dmitry Fleischman, Randy K. Schwartz, and the Georgia

Southern problem solvers submitted solutions. We follow the approach of the "Eagle Solvers" from Georgia for part two, which provides the answer for part one as well.

The 180° rotation of the square $[1,n]\times[1,n]$ takes the point (i,j) to the point $(n+1-i, n+1-j)$. Thus a rotation-invariant permutation p must satisfy $p(n+1-i)=n+1-p(i)$. This means that the values $p(1),\ p(2),\ \ldots,\ p(\lfloor n/2\rfloor)$ determine the values of $p(k)$ for $k>(n+1)/2$; and when n is odd, we must have $p((n+1)/2)=(n+1)/2$. Thus, there are $2\lfloor n/2\rfloor$ possible values for $p(1)$ (any value from 1 to n for n even, and any of these values except $(n+1)/2$ for n odd). A choice j for $p(1)$ eliminates both j and $p(n)=n+1-j$ as choices for $p(2)$, so there are two fewer options for $p(2)$, and so on. Hence the total number of rotation-invariant permutations is

$$2\left\lfloor\frac{n}{2}\right\rfloor\left(2\left\lfloor\frac{n}{2}\right\rfloor-2\right)\left(2\left\lfloor\frac{n}{2}\right\rfloor-4\right)\cdots=\left(2\left\lfloor\frac{n}{2}\right\rfloor\right)!!.$$

This means there are $6!!=6\cdot4\cdot2=\mathbf{48}$ rotation-invariant permutations on six objects.

Binomial Determinant (P433). This problem was submitted by George Stoica (Saint John, Canada). Let A_n be the $n\times n$ matrix where the entry in row i and column j is given by the binomial coefficient

$$\binom{n+1}{2i-j},$$

with the convention that this coefficient has value 0 when $2i-j$ is negative or larger than $n+1$. Show

$$\det A_n = 2^{n(n+1)/2}=2^{\binom{n+1}{2}}.$$

A numerical solution establishing the result for $n\le 5$ was submitted by Dmitry Fleischman. We present the general argument from the proposer.

Let B be the $n\times n$ lower triangular matrix of ones, which has determinant one, and check that the entries of B^{-1} are $b_{i,i}=1$, $b_{i+1,i}=-1$, and $b_{i,j}=0$ otherwise. We claim that for $n\ge 2$, BA_nB^{-1} is a matrix with the block form

$$BA_nB^{-1}=\begin{bmatrix} A_{n-1} & \mathbf{v} \\ \mathbf{0} & 2^n \end{bmatrix},$$

where $\mathbf{v}$ is an irrelevant $(n-1)\times 1$ column vector.

Because $A_1=[2]$, the result follows by induction. Multiplying out on the left-hand side, the claim only requires that for $i\le n$ and $j<n$,

$$\sum_{k=1}^{i}\left[\binom{n+1}{2k-j}-\binom{n+1}{2k-j-1}\right]=\binom{n}{2i-j},\ \text{and}\ \sum_{k=1}^{n}\binom{n+1}{2k-n}=2^n.$$

However, these facts are immediately implied by the well-known binomial identities

$$\sum_{k=0}^{n}(-1)^k\binom{r}{k}=(-1)^n\binom{r-1}{n}$$

$$\text{and}\ \sum_i\binom{n}{2i}=\sum_i\binom{n}{2i+1}=2^{n-1}.$$

CAROUSEL SOLUTION

Suppose $g(t)=t$ for some integer t. Then a, b, and t are distinct because $g(x)$ swaps a and b and fixes t. Observe that $g(x)-g(y)=(x-y)Q(x,y)$ where $Q(x,y)$ is a polynomial with integer coefficients. Now $(a-b)Q(a,b)=b-a$; $(a-t)Q(a,t)=b-t$; and $(b-t)Q(b,t)=a-t$. Multiplying these quantities results in $Q(a,b)Q(a,t)Q(b,t)=-1$, implying $Q(a,t)=\pm1$. If $Q(a,t)=1$, then $a-t=b-t$ so $a=b$, which is a contradiction. Hence, $Q(a,t)=-1$ so that $-(a-t)=b-t$ or, equivalently, $2t=a+b$.

Let $g_1(x)=-x+a+b$. Check that g_1 swaps a and b and fixes $t=(a+b)/2$. Next, observe that $g_2(x)=-x+(a+b)+(x-a)(x-b)$ swaps a and b. Moreover, $g_2((a+b)/2)=(a+b)/2-(a-b)^2/4$. Thus, g_2 has no integer fixed point.

This proposition appeared in an unpublished manuscript by Gerd Baron in 1991. Special thanks to Anthony Bevelacqua for sharing the result with the editor.

SUBMISSION AND CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to *MHproblems@maa.org* and *MHsolutions@maa.org*, respectively (PDF format preferred). Paper submissions can be sent to Jeremiah Bartz, UND Math Dept., Witmer Hall 313, 101 Cornell St. Stop 8376, Grand Forks, ND 58202-8376. Please include your name, email address, and affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is October 31, 2022.

10.1080/10724117.2022.2092341