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Problems and Solutions

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

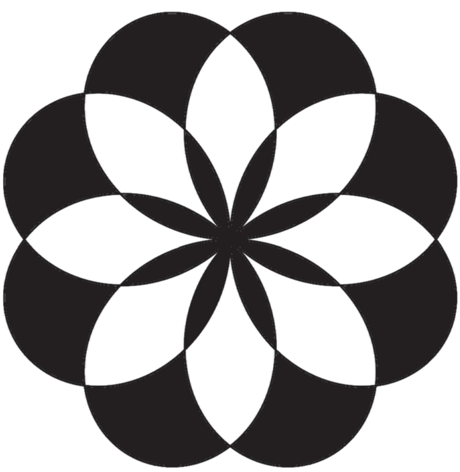
Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

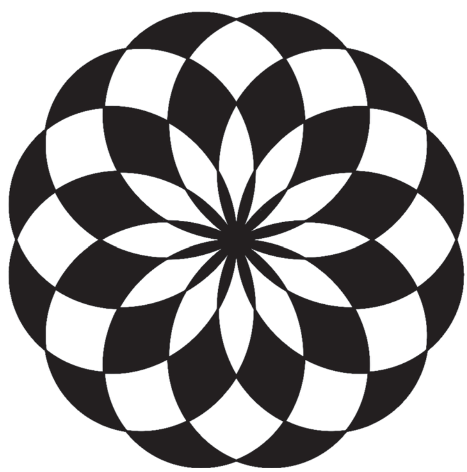
Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than September 15, 2022. Sending both pdf and TEXfiles is ideal.

PROBLEMS

1221. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA.*

Shown below (from left to right) are the graphs of $r = \sin 4\theta/3$ and $r = \sin 6\theta/5$, where every other adjacent region (starting from the outside) is shaded black. Find the total shaded area for any such graph $r = \sin(k+1)\theta/k$, where $k > 0$ is an odd integer and θ ranges from 0 to $2k\pi$.





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1222. *Proposed by Kent Holing, Trondheim, Norway.*

Consider the parabola $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ with real coefficients, $B \neq 0$ and $A, C > 0$.

1. Show that the parabola is non-degenerate if and only if $\beta = BE - CD \neq 0$.
2. Show that in the degenerate case, the parabola can be given by the formula $f(x, y) = Ax + By + D \pm \sqrt{\alpha_1} = 0$ for $\alpha_1 = D^2 - AF$ or (equivalently) by $f(x, y) = Bx + Cy + E \pm \sqrt{\alpha_2} = 0$ for $\alpha_2 = E^2 - CF$ and $\alpha_{1,2} \geq 0$.
3. When $\beta \neq 0$, show that $(A + C)(Bx_T + Cy_T) + BD + CE = 0$ for the coordinates x_T and y_T of the vertex T.
4. Using 3, show that $x_T = -\frac{\alpha_2}{2\beta} + At$ for $t = \frac{\beta}{2C(A+C)^2}$.
5. Show that the coordinates of the focus F of the parabola are $x_F = x_T + Ct$ and $y_F = y_T - Bt$.

1223. *Don Redmond, Southern Illinois University, Carbondale, IL.*

Let h be a positive integer and define the nth rectangular number of order h, denoted by $R_h(n)$, as $R_h(n) = n(n + h)$. Determine all positive integers h for which the equation $R_h(n) = m^2$ has a solution for some positive integers n and m.

1224. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let G be a finite group, and suppose that for any subgroups H and K of G, we have $|H \cap K| = \gcd(|H|, |K|)$. Prove that G is cyclic.

1225. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

All rings R throughout are commutative with $1 \neq 0$ and all subrings S of R are unital (that is, $1 \in S$). Recall that a ring R is *chained* provided that for any ideals I and J of R, either $I \subseteq J$ or $J \subseteq I$.

1. Give an example of a ring R which is not a field with the property that every subring of R is chained.
2. Suppose now that R is *reduced*, that is, R has no nonzero nilpotents. Prove that if every subring of R is chained, then R is a field.

Polynomials of degree n tangent to a circle at $n-1$ points

1196. *Proposed by Ferenc Beleznay, Mathleaks, Budapest, Hungary, and Daniel Hwang, Wuhan Britain-China School, Wuhan, China.*

Prove or disprove: for every positive integer n, there exists a polynomial of degree $n+1$ with real coefficients whose graph is tangent to some circle at n points.

Solution by Mark Wildon, Royal Holloway, Egham, UK.

Such polynomials exist. Shifting n, we shall prove that for each $n \in \mathbb{N}$ with $n \geq 3$ there exists a polynomial P_n of degree n with coefficients in the integers such that the graph of $P_n(x)$ is tangent to the unit circle at exactly $n-1$ points in the open interval $(-1, 1)$. For $n = 2$ we may simply take $P_2(x) = 1$, which is tangent to the unit circle at 0 and has degree 0.

To define the P_n for $n \geq 3$, we need the Chebyshev polynomials of the second kind. Recall that, in the usual notation, U_m is the unique polynomial with real coefficients of degree m such that $(\sin\theta)U_m(\cos\theta) = \sin(m+1)\theta$. For instance $U_0(x) = 1$, $U_1(x) = 2x$, and since $\sin 3\theta = -\sin^3\theta + 3\sin\theta\cos^2\theta = \sin\theta(-\sin^2\theta + 3\cos^2\theta) = \sin\theta(-1 + 4\cos^2\theta)$ we have $U_2(x) = 4x^2 - 1$. In fact each U_n has integer coefficients. For each $n \in \mathbb{N}$ with $n \geq 4$, define

$$P_n(x) = x^2 U_{n-2}(x) - 2xU_{n-3} + U_{n-4}.$$

As shown in [1, Theorem 5], the defining property of U_m and the relation $2\cos\theta\sin r\theta = \sin(r+1)\theta + \sin(r-1)\theta$ imply that if $n \geq 4$ then

$$\begin{aligned}
&(\sin\theta)P_n(\cos\theta)\\
&= (\cos^2\theta\sin\theta)U_{n-2}(\cos\theta) - 2(\cos\theta\sin\theta)U_{n-3}(\cos\theta) + (\sin\theta)U_{n-4}(\cos\theta)\\
&= \cos^2\theta\sin(n-1)\theta - 2\cos\theta\sin(n-2)\theta + \sin(n-3)\theta\\
&= (1-\sin^2\theta)\sin(n-1)\theta - \sin(n-1)\theta - \sin(n-3)\theta + \sin(n-3)\theta\\
&= -\sin^2\theta\sin(n-1)\theta.
\end{aligned}$$

Hence, $P_n(\cos\theta) = -\sin\theta\sin(n-1)\theta$ for each such n. Setting $P_3(x) = 2x^3 - 2x$ we have $P_3(\cos\theta) = 2\cos^3\theta - 2\cos\theta = 2(\cos^2\theta - 1)\cos\theta = -2\sin^2\theta\cos\theta = -\sin\theta\sin 2\theta$. Therefore,

$$P_n(\cos\theta) = -\sin\theta\sin(n-1)\theta \quad \text{if } n \geq 3. \tag{$\star$}$$

Since each U_m has integer coefficients, so does each P_n.

Observe that, by $(\star)$,

$$(\cos\theta)^2 + P_n(\cos\theta)^2 = \cos^2\theta + \sin^2\theta\sin^2(n-1)\theta \leq \cos^2\theta + \sin^2\theta = 1.$$

Hence, the graph of $P_n(x)$ for $-1 \leq x \leq 1$ lies inside the closed unit disc. Moreover, we have $(\cos\theta)^2 + P_n(\cos\theta)^2 = 1$ if and only if $\sin^2(n-1)\theta = 1$, so if and only if $\theta = \frac{(2k-1)\pi}{n-1}$ for some $k \in \mathbb{N}$. Thus if $x = \cos\frac{(2k-1)\pi}{n-1}$ and $x \in (-1, 1)$, the graph of $P_n(x)$ is tangent to the unit circle.

To get distinct values of $\cos\theta$, we may assume that $\theta \in [0, \pi]$. If $n = 2m$ is even then there are $2m - 1$ distinct tangent points, obtained by taking $k = 1, \ldots, m - 1, m, m + 1, \ldots 2m - 1$ to get x-coordinates

$$\cos\frac{\pi}{2m-1}, \ldots, \cos\frac{(2m-3)\pi}{2m-1}, \cos\frac{(2m-1)\pi}{2m-1} = -1, -\cos\frac{2\pi}{2m-1}, \ldots, -\cos\frac{(2m-2)\pi}{2m-1}.$$

If $n = 2m + 1$ is odd, then there are $2m$ distinct tangent points, obtained by taking $k = 1, \ldots, m - 1, m$ to get x coordinates

$$\cos\frac{\pi}{2m}, \ldots, \cos\frac{(2m-3)\pi}{2m}, \cos\frac{(2m-1)\pi}{2m}$$

and then $k = m + 1, \ldots, 2m$ to get x coordinates

$$-\cos\frac{\pi}{2m}, \ldots, -\cos\frac{(2m-3)\pi}{2m}, -\cos\frac{(2m-1)\pi}{2m}.$$

This completes the proof.

Remark. We remark that since $P_n(1) = P_n(\cos 0) = 0$ and $P_n(-1) = P_n(\cos\pi) = 0$ by $(\star)$, the graph of $P_n(x)$ meets the graph of the unit circle at $x = \pm 1$; of course since the unit circle has a vertical asymptote at these points, the graph is not tangent. Thus, P_n is tangent to the unit circle at $n - 1$ points and has two further intersection points. Since tangent points have multiplicity (at least) 2, this meets the bound in Bezout's Theorem, that the intersection multiplicity between the algebraic curves $y = P_n(x)$ and $x^2 + y^2 = 1$ of degrees n and 2, respectively, is $2n$, and shows that each tangent point has degree exactly 2.

References

[1] Janjić, M. (2008). On a class of polynomials with integer coefficients. *J. Integer Seq.* 11(5): Article 08.5.2, 9.

Also solved by the proposer. We received one incomplete solution.

Matrices with presistently unequal rows

1197. *Proposed by Valery Karachik and Leonid Menikhes, South Ural State University, Chelyabinsk, Russia*

Let A be an arbitrary $n \times m$ matrix that has no equal rows. Find a necessary sufficient condition relating n and m so that there exists a column of A, after removal of which, all rows remain different.

Solution by Eugene Herman, Grinnell College, Grinnell, Iowa.

The given property holds in a trivial sense when $n = 1$ or $m = 1$. In both cases, after a column has been removed there do not exist two rows that are equal. Otherwise, the necessary and sufficient condition is $2 \leq n \leq m$. Suppose first that $m + 1 = n \geq 2$. Let $A = [a_{ij}]$, where $a_{ij} = 0$ when $j \geq i$ and $a_{ij} = 1$ when $j < i$. If column j of A is removed then rows j and $j + 1$ are equal; hence the given property fails to hold. If $n \geq m + 2$, construct the first $m + 1$ rows of A as before and fill in the rest of the matrix so all rows are different.

Suppose $2 \leq n \leq m$ and suppose the given property does not hold. Thus, for each $j \in \{1, 2, \ldots, m\}$, there exists a pair of rows $P_j = \{r, s\}$ such that r and s are unequal but become equal when the jth entry is removed from each. We create an undirected graph as follows. Each vertex corresponds to a row, and so the number of vertices is n. The edges correspond to the sets P_j; specifically, (r, s) is an edge if and only if $\{r, s\} = P_j$ for some j. Hence the number of edges is m. No vertex is joined to itself by an edge and no two vertices are joined by more than one edge. We show that the graph contains no cycles. Suppose $(r_1, \ldots, r_k)$ is a cycle; that is, $r_1, \ldots, r_k$ are distinct vertices and $(r_1, r_2), \ldots, (r_{k-1}, r_k), (r_k, r_1)$ are edges. The edges correspond to different columns, which we may assume are columns 1 through k (by permuting columns, if necessary). Let $r_1 = (a_1, a_2, \ldots, a_m)$. Thus, $r_2 = (b_1, a_2, a_3, \ldots, a_m)$ where $b_1 \neq a_1$ and $r_3 = (b_1, b_2, a_3, \ldots, a_m)$ where $b_2 \neq a_2$, and so on until $r_k = (b_1, b_2, \ldots, b_{k-1}, a_k, \ldots, a_n)$ where $b_{k-1} \neq a_{k-1}$. Then (r_k, r_1) cannot be an edge since r_k and r_1 differ in in $k - 1$ entries and $k - 1 > 1$. Our graph is therefore a tree. In a tree, the number of vertices is always larger than the number of edges, and so $m < n$. This contradiction establishes our necessary and sufficient condition.

Also solved by the proposer.

The cardinality of a set of maximal ideals

1198. *Proposed by Alan Loper, The Ohio State University, Newark OH, and Greg Oman, The University of Colorado, Colorado Springs, CO.*

Let n be a nonnegative integer, and consider the ring $R := \mathbb{Q}[X_0, \ldots, X_n]$ of polynomials (via usual polynomial addition and multiplication) in the (commuting) variables $X_0, \ldots X_n$ with coefficients in $\mathbb{Q}$. It is well known that R is a Noetherian ring, and so every ideal of R is finitely generated. Since R is countable, and there are but countably many finite subsets of a countable set, we deduce that R has but countably many ideals and thus, in particular, countably many maximal ideals. Next, let $X_0, X_1, X_2, \ldots$ be a countably infinite collection of indeterminates. Observe that (to within isomorphism) $\mathbb{Q}[X_0] \subseteq \mathbb{Q}[X_0, X_1] \subseteq \mathbb{Q}[X_0, X_1, X_2] \subseteq \cdots$. Let $\mathbb{Q}[X_0, X_1, X_2, \ldots]$ be the union of the this increasing chain. How many maximal ideals does the ring $\mathbb{Q}[X_0, X_1, X_2, \ldots]$ have? (More precisely, what is the cardinality of the set of maximal ideals of $\mathbb{Q}[X_0, X_1, X_2, \ldots]$?)

Solution by Kenneth Schilling, University of Michigan-Flint, Flint, Michigan.

Since $\mathbb{Q}\,[X_0, X_1, X_2....]$ has countably many elements, it has at most $2^{\aleph_0}$ maximal ideals. We shall exhibit $2^{\aleph_0}$ maximal ideals, proving that this is the exact cardinality.

Let $p_0(t) = t$ and $p_1(t) = t - 1$. For each infinite sequence $\alpha : \mathbb{N} \to \{0, 1\}$, let I_α be the ideal of $\mathbb{Q}\,[X_0, X_1, X_2....]$ generated by the set of polynomials

$$\{p_{\alpha(k)}(X_k) : k = 1, 2, 3, ...\}.$$

Since $p_{\alpha(k)}(\alpha(k)) = 0$, for any $q\,(X_1, X_2, ..., X_n) \in I_\alpha$,

$$q\,(\alpha(0), \alpha(1), ..., \alpha(n)) = 0.$$

It follows that I_α is a proper ideal of $\mathbb{Q}\,[X_0, X_1, X_2....]$, and so is contained in a maximal ideal M_α.

Now consider any pair α, β of distinct infinite sequences from $\{0, 1\}$. For some k, $\{\alpha(k), \beta(k)\} = \{0, 1\}$, so $\{p_{\alpha(k)}(X_k), p_{\beta(k)}(X_k)\} = \{X_k, X_k - 1\}$. Therefore the ideal generated by $I_\alpha \cup I_\beta$ is the whole ring $\mathbb{Q}\,[X_0, X_1, X_2....]$. It follows that the union $M_\alpha \cup M_\beta$ of maximal ideals must also generate the whole ring, and so, in particular, $M_\alpha \neq M_\beta$.

We conclude that the set of ideals M_α over all infinite sequences $\alpha : \mathbb{N} \to \{0, 1\}$ is of cardinality $2^{\aleph_0}$, and the proof is complete.

Also solved by Paul Budney, Sunderland, MA; and the proposer. We received one incomplete solution.

An oscillating function with prescribed zeros

1199. *Proposed by Corey Shanbrom, Sacramento State University, Sacramento, CA.*

Find a smooth, oscillating function whose periods form a bi-infinite geometric sequence. More precisely, given a positive $\lambda \neq 1$, find a smooth function f on an open half-line whose root set $\mathcal{R}$ is given by

$$\mathcal{R} = \left\{ \cdots - \frac{1}{\lambda^3} - \frac{1}{\lambda^2} - \frac{1}{\lambda}. - \frac{1}{\lambda^2} - \frac{1}{\lambda}, -\frac{1}{\lambda}, 0, \right.$$
$$\left. 1, 1+\lambda, 1+\lambda+\lambda^2, 1+\lambda+\lambda^2+\lambda^3, \cdots \right\}.$$

Editor's note: The problem statement in the March 2021 issue omitted one of the zeros. The functions defined in the submitted solutions included this value in their root set.

Solution by Albert Natian, Los Angeles Valley College, Valley Glen, California..

Answer: $f(x) = \sin\left(\frac{\pi \ln[(\lambda-1)x+1]}{\ln \lambda}\right)$ defined on $\left([1-\lambda]^{-1}, \infty\right)$ if $\lambda > 1$ and defined on $\left(-\infty, [1-\lambda]^{-1}\right)$ if $\lambda < 1$.

Justification It's clear that $\sin\theta = 0 \iff \theta = n\pi,\ n \in \mathbb{Z}$. So

$$\begin{aligned}
f(x) = 0 &\iff \sin\left(\frac{\pi \ln[(\lambda-1)x+1]}{\ln\lambda}\right) = 0 \\
&\iff \frac{\pi \ln[(\lambda-1)x+1]}{\ln\lambda} = n\pi,\ n \in \mathbb{Z} \\
&\iff \ln[(\lambda-1)x+1] = n\ln\lambda,\ n \in \mathbb{Z} \\
&\iff \ln[(\lambda-1)x+1] = \ln\lambda^n,\ n \in \mathbb{Z} \\
&\iff (\lambda-1)x+1 = \lambda^n,\ n \in \mathbb{Z} \\
&\iff x = \frac{\lambda^n - 1}{\lambda - 1} \text{ if } n \geq 0,\ x = -\frac{1}{\lambda} \cdot \frac{\left(\frac{1}{\lambda}\right)^{-n} - 1}{\left(\frac{1}{\lambda}\right) - 1} \text{ if } n < 0,\ n \in \mathbb{Z} \\
&\iff x = \sum_{j=0}^{n-1} \lambda^j \text{ if } n \geq 0,\ x = -\sum_{j=1}^{-n} \left(\frac{1}{\lambda}\right)^j \text{ if } n < 0,\ n \in \mathbb{Z}.
\end{aligned}$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

A recurrence satisfied by a sequence with a given generating function

1200. *Proposed by Russ Gordon, Whitman College, Walla Walla, Washington, and George Stoica, St. John, New Brunswick, Canada*

Let c be an arbitrary real number. Prove that the sequence $(a_n)_{n\geq 0}$ defined by

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - cx + cx^2 - x^3}$$

satisfies $a_n(a_n - 1) = a_{n+1}a_{n-1}$ for all $n \geq 1$.

Solution 1 by Michel Bataille, Rouen, France.

Since $1 - cx + cx^2 - x^3 = (1 - x)(1 + (1 - c)x + x^2)$, the sequence (a_n) is the unique sequence satisfying

$$(1 + (1 - c)x + x^2) \cdot \sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n.$$

Multiplying out on the left, we obtain $a_0 = 1,\ a_1 + (1 - c)a_0 = 1$ and for $n \geq 2$

$$a_n + a_{n-1}(1 - c) + a_{n-2} = 1. \tag{1}$$

Now, we prove that $a_n(a_n - 1) = a_{n+1}a_{n-1}$ for all $n \geq 1$ by induction.
Since $a_1(a_1 - 1) = c(c - 1)$ and (using (1)), $a_2 a_0 = a_2 = 1 - a_1(1 - c) - a_0 = c(c - 1)$, the relation holds for $n = 1$.
Assume that $a_n(a_n - 1) = a_{n+1}a_{n-1}$ for some integer $n \geq 1$. Then, we have

$$\begin{aligned}
a_n a_{n+2} &= a_n(1 - a_n - (1 - c)a_{n+1}) \quad \text{(using (1))}\\
&= a_n(1 - a_n) - a_n a_{n+1}(1 - c)\\
&= -a_{n+1}a_{n-1} - a_n a_{n+1}(1 - c) \quad \text{(by assumption)}\\
&= -a_{n+1}(a_{n-1} + a_n(1 - c))\\
&= -a_{n+1}(1 - a_{n+1}) \quad \text{(using (1))},
\end{aligned}$$

hence $a_{n+1}(a_{n+1} - 1) = a_n a_{n+2}$. This completes the induction step and the proof.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

Denote the recurrence relation $a_n\,(a_n - 1) = a_{n+1}a_{n-1}$ by *.

- If $c = -1$, then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{4(1 - x)} + \frac{1}{4(1 + x)} + \frac{1}{2(1 + x)^2},$$

  so that $a_n = \frac{1}{4}\,[1 + (-1)^n(2n + 3)]$, and * holds.
- If $c = 3$, then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1 - x)^3},$$

  so that $a_n = \frac{(n+1)(n+2)}{2}$, and * again holds.

In what follows, we assume that $c \neq -1, 3$. Let $\alpha = \frac{c-1+\sqrt{(c-3)(c+1)}}{2}$, so that $\alpha \neq -1, 0, 1$. We have $c = \dfrac{1+\alpha+\alpha^2}{\alpha}$, and

$$\frac{1}{1-cx+cx^2-x^3} = \frac{\alpha}{(1-x)(\alpha-x)(1-\alpha x)}$$
$$= \frac{\alpha}{(1-\alpha)^2}\left(\frac{1}{(1+\alpha)(\alpha-x)} + \frac{\alpha^2}{(1+\alpha)(1-\alpha x)} - \frac{1}{1-x}\right).$$

Hence

$$a_n = \frac{\alpha}{(1-\alpha)^2}\left(\frac{1}{(1+\alpha)\alpha^{n+1}} + \frac{\alpha^{n+2}}{1+\alpha} - 1\right) = \frac{\left(1-\alpha^{n+1}\right)\left(1-\alpha^{n+2}\right)}{(1+\alpha)\left(1-\alpha\right)^2\alpha^n},$$

and it is easy to check that * holds in this case as well.

Solution 3 by Graham Lord, Princeton, New Jersey.

That $a_0 = 1$ is immediate from the substitution $x = 0$ in the equation. The latter's first and second derivatives at 0 show $a_1 = c$ and $a_2 = c(c-1)$, respectively. Note, $c - 1 = \frac{a_2+a_0-1}{a_1}$ and $a_1(a_1-1) = a_2a_0$. For convenience, set $a_{-1} = 0$, so $a_0(a_0-1) = a_1a_{-1}$.

The equation's RHS denominator, $1-cx+cx^2-x^3$ factors into $(1-x)$ and $(1-(c-1)x+x^2)$. So multiplication of the equation through by the latter factor gives: $1 + \sum_{n=1}^{\infty}(a_n - (c-1)a_{n-1} + a_{n-2})x^n = \frac{1}{1-x} = 1 + x + x^2 + \ldots$.

Hence for all $n \geq 1$, as the coefficients of x^n on both sides of this last equation are equal: $(a_n - (c-1)a_{n-1} + a_{n-2}) = 1$. Equivalently: $c - 1 = \frac{a_n+a_{n-2}-1}{a_{n-1}}$. That is, for any $n \geq 1$ the ratio, $\frac{a_n+a_{n-2}-1}{a_{n-1}}$ is constant, independent of n, and equal to $c-1$. In particular: $\frac{a_n+a_{n-2}-1}{a_{n-1}} = \frac{a_{n+1}+a_{n-1}-1}{a_n}$. The latter simplified is the sought after identity $a_n(a_n-1) = a_{n+1}a_{n-1}$.

Also solved by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule, Mittelhessen, Germany; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Kyle Calderhead, Malone U.; Hongwei Chen, Christopher Newport U.; FAU Problem Solving Group, Florida Atlantic U.; Geuseppe Fera, Vicenza, Italy; Dmitry Fleischman, Santa Monica, CA; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus (jointly); G. C. Greubel, Newport News, VA; GWstat Problem Solving Group, The George Washington U.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Omran Kouba, Higher Inst. for Applied Sci. and Tech., Damascus, Syria. Northwestern U. Math Problem Solving Group; Carlos Shine, São Paulo, Brazil; Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lake Forest C.; Michael Vowe, Therwil, Switzerland; and the proposer.