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Problems and Solutions

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than November 15, 2022. Sending both pdf and TEXfiles is ideal.

PROBLEMS

1226. *Proposed by George Apostolopoulos, Messolongi, Greece.*

Let a, b, and c be positive real numbers. Prove that $\ln \frac{27abc}{(a+b+c)^3} \leq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3}$.

1227. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Do there exist functions $f\colon (0,1) \to \mathbb{R}$ and $g\colon (0,1) \to \mathbb{R}$ such that for all x and $y \in (0,1)$, the following two conditions are satisfied:

1. $f(x) < g(x)$, and
2. if $x < y$, then $g(x) < f(y)$?

Either find examples of such f and g or prove that no such f and g exist.

1228. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a ring, and let $f\colon R \to R$ be a function. Say that f is *multiplicative* if $f(xy) = f(x)f(y)$, $f(0) = 0$, and (if R has an identity) $f(1) = 1$. Find all commutative rings R (not assumed to have an identity) with the following two properties:

1. There exists an element $a \in R$ which is not nilpotent, and
2. every multiplicative map $f : R \to R$ is either the identity map or the zero map.

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1229. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ii} = 0$ and $a_{ij} = b_i c_j$ for $i \neq j$, where $b_i > 0$ and $c_j \geq 0$ for $1 \leq i, j \leq n$. Prove that the spectral radius of A is strictly less than 1 if and only if $\sum_{i=1}^{n} \frac{b_i c_i}{b_i c_i + 1} < 1$.

1230. *Proposed by Jason Zimba, Amplify, New York, NY.*

A Heronian triangle is a triangle with positive integer side lengths and positive integer area. Denoting the side lengths of a Heronian triangle by a, b, and c, the triangle is called *primitive* if $\gcd(a, b, c) = 1$. We shall say that a primitive Heronian triangle *has an equivalent rectangle* if there exists a rectangle with integer length and width that shares the same perimeter and area as the triangle. Show that infinitely many primitive Heronian triangles have equivalent rectangles.

The intersection of an ellipsoid and a plane

1201. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Consider the ellipsoid $\frac{x^2}{4}+y^2+z^2=1$ and the ellipse E which is the intersection of the ellipsoid with the plane $ax+by+cz=0$, where $P=(a,b,c)$ is a random point on the unit sphere (so $a^2+b^2+c^2=1$). Now consider the random variable A_E, the area of the ellipse E. If the point P is chosen on the unit sphere with uniform distribution with respect to the area, what is the expectation of A_E?

Solution by Bruc Burdick, Providence, RI.

Since the ellipsoid has rotational symmetry about the x-axis, the plane could be rotated around the x-axis without changing the size or shape of E. The plane could be so rotated until it contained the y-axis, for example. Then, unless the plane were the yz-plane, its equation could be put in the form $z=mx$. It is then evident from the equations $\frac{x^2}{4}+y^2+z^2=1$ and $z=mx$ that the lines

$$x=z=0$$

and

$$z=mx,\ y=0$$

would be the lines of symmetry of E.

The major and minor axes of E must lie on these lines. It follows that, before the rotation, one of these axes lay in the yz-plane and, therefore, its endpoints were $\frac{1}{\sqrt{1-a^2}}(0,c,-b)$ and $\frac{1}{\sqrt{1-a^2}}(0,-c,b)$. The other axis would lie in the direction of the cross product of $c\mathbf{j}-b\mathbf{k}$ and $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, which is $(1-a^2)\mathbf{i}-ab\mathbf{j}-ac\mathbf{k}$. So, the endpoints of that axis would be given by

$$x=\pm\frac{2\sqrt{1-a^2}}{\sqrt{1+3a^2}}$$

$$y=\mp\frac{2ab}{\sqrt{1-a^2}\sqrt{1+3a^2}}$$

$$z=\mp\frac{2ac}{\sqrt{1-a^2}\sqrt{1+3a^2}}.$$

It then is apparent that the lengths of the semiminor axis and the semimajor axis must be 1 and $\frac{2}{\sqrt{1+3a^2}}$, respectively. Therefore,

$$A_E=\frac{2\pi}{\sqrt{1+3a^2}}.$$

We put P in spherical coordinates with the x-axis as the pole. So, we have $a=\cos\phi$, $b=\sin\phi\cos\theta$, and $c=\sin\phi\sin\theta$. Then the surface integral of A_E is

$$\int_0^{\pi}\int_0^{2\pi}\frac{2\pi}{\sqrt{1+3\cos^2\phi}}\sin\phi\, d\theta\, d\phi=4\pi^2\int_0^{\pi}\frac{1}{\sqrt{1+3\cos^2\phi}}\sin\phi\, d\phi.$$

With the substitution $u = \sqrt{3}\cos\phi$, $du = -\sqrt{3}\sin\phi d\phi$, we have

$$\int_0^{\pi}\int_0^{2\pi} \frac{2\pi}{\sqrt{1+3\cos^2\phi}} \sin\phi d\theta d\phi = \frac{4\pi^2}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{\sqrt{1+u^2}}.$$

Standard trigonometric substitution leads to

$$\begin{aligned}
\int_0^{\pi}\int_0^{2\pi} \frac{2\pi}{\sqrt{1+3\cos^2\phi}} \sin\phi d\theta d\phi &= \frac{4\pi^2}{\sqrt{3}} \ln|\sec v + \tan v|\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
&= \frac{4\pi^2}{\sqrt{3}} \ln\frac{2+\sqrt{3}}{2-\sqrt{3}} \\
&= \frac{4\pi^2}{\sqrt{3}} \ln(2+\sqrt{3})^2 \\
&= \frac{8\pi^2}{\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

To find the expected value, we must now divide by the area of the unit sphere, which is 4π. So,

$$E(A_E) = \frac{2\pi}{\sqrt{3}} \ln(2+\sqrt{3}).$$

Also solved by EAGLE PROBLEM SOLVERS, Georgia Southern U.; JOHN FITCH, Minnetonka, MN; KELLY MCLENITHAN, Los Alamos, NM: DIDIER PINCHON, Toulouse, France; RANDY SCHWARTZ, Schoolcraft C. (ret.); and the proposer. We received two incorrect solutions.

A condition on two square matrices forcing the adjugate of one of them two be zero

1202. *Proposed by Cezar Lupu, Texas Tech University, Lubbock, TX.*

Let A and B be $n \times n$ matrices with complex entries such that B is not invertible and $AB^2 - B^2A = \mathrm{adj}(B)$. Prove that $(\mathrm{adj}(B))^2 = O_n$, the $n \times n$ zero matrix. Here, $\mathrm{adj}(B)$ is the adjugate matrix of B.

Solution by Michel Bataille, Rouen, France.

Since B is not invertible, we have $\mathrm{rank}(B) \le n-1$. If $\mathrm{rank}(B) \le n-2$. all the cofactors of B are 0, hence, $\mathrm{adj}(B) = O_n$ and therefore $(\mathrm{adj}(B))^2 = O_n$.
From now on, we suppose that $\mathrm{rank}(B) = n-1$. Then, $\dim(\ker B) = 1$ and since

$$B \cdot \mathrm{adj}(B) = (\det(B))I_n = O_n,$$

the columns of $\mathrm{adj}(B)$ are in $\ker B$. It follows that $\mathrm{rank}(\mathrm{adj}(B)) \le 1$; however, $\mathrm{adj}(B) \ne O_n$ (one of the cofactor of B being nonzero), hence, $\mathrm{rank}(\mathrm{adj}(B)) = 1$.
Now, if M is a $n \times n$ matrix of rank 1, then $M^2 = \mathrm{tr}(M)M$ where $\mathrm{tr}(M)$ is the trace of M [*Proof*: $M = UV^T$ for some column vectors U, V, hence, $M^2 = U(V^TU)V^T = (V^TU)UV^T = \mathrm{tr}(M)M$]. It follows that

$$(\mathrm{adj}(B))^2 = \mathrm{tr}(\mathrm{adj}(B))\mathrm{adj}(B) = O_n,$$

the latter equality because

$$\mathrm{tr}(\mathrm{adj}(B)) = \mathrm{tr}(AB^2 - B^2A) = \mathrm{tr}(AB^2) - \mathrm{tr}(B^2A) = \mathrm{tr}(AB^2) - \mathrm{tr}(AB^2) = 0,$$

(since $\mathrm{tr}(MN) = \mathrm{tr}(NM)$).

Also solved by Eugene Herman, Grinnell C.; Koopa Tak Lun Koo, Hong Kong STEAM Academy; Didier Pinchon, Toulouse, France; Albert Stadler, Herrliberg, Switzerland; Jeffrey Stuart, Pacific Luthern U.; and the proposer.

Sum-free sets

1203. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let $X \subseteq \mathbb{Z}$ and let $k > 1$ be an integer. Say that X is k *sum-free* if the sum of any k elements of X (with repetitions allowed) is *not* in X.

1. Let $n > 1$ be an integer. Prove that there is an infinite $X \subseteq \mathbb{Z}$ which is k sum-free for $1 < k \leq n$.
2. Is there an infinite $X \subseteq \mathbb{Z}$ which is k sum-free for every integer $k > 1$?

Solution by Eagle Problem Solvers, Georgia Southern University.

1. For each integer $n > 1$, let $Y_n = \{x \in \mathbb{Z} : x \equiv 1 \pmod{n}\}$. Then if $1 < k \leq n$ is an integer, then the sum of any k elements in Y_n is congruent to $k \pmod{n}$, which is not congruent to $1 \pmod{n}$. Thus, Y_n is k sum-free for every integer k with $1 < k \leq n$.
2. We prove that there is no infinite subset of $\mathbb{Z}$ which is k sum-free for every integer $k > 1$. Suppose that X is such a set. Since X is infinite, then either $X \cap \mathbb{Z}^+$ or $X \cap \mathbb{Z}^-$ must be infinite; without loss of generality, suppose $X^+ = X \cap \mathbb{Z}^+$ is infinite. Enumerate the elements of X^+ so that $X^+ = \{x_j : j \in \mathbb{N}\}$, where $x_1 < x_2 < \cdots$, and let $n = x_1$.

   Lemma 1. *If x_i and x_j are distinct elements of X^+, then $x_i \not\equiv x_j \pmod{n}$.*

   Proof. Suppose, without loss of generality, that $x_i < x_j$. If $x_j \equiv x_i \pmod{n}$, then $x_j - x_i = qn$, where $q \in \mathbb{N}$, so that $x_j = x_i + qx_1$ is a $q + 1$ sum of elements of X^+, with $q + 1 > 1$. ■

   By the lemma and the Division Algorithm, $x_2 = q_2 n + r_2$ where q_2 and r_2 are positive integers and $0 < r_2 < n$. Similarly, by the applying the lemma again, $x_3 = q_3 n + r_3$ is not congruent to 0 or r_2 modulo n. By repeating this process, $\{0, r_2, r_3, \ldots, r_n, r_{n+1}\}$ forms a set of $n + 1$ distinct congruence classes modulo n, which is impossible.

Also solved by Paul Budney, Sunderland, MA; Eugene Herman, Grinnell C.; Isaiah Adams, Luke Loprieno, Rachel McMullan, Elizabeth Mike (students), and David Schmitz, North Central C.; Northwestern U. Math Problem Solving Group; Harvey Perkins, Cuesta C.; Didier Pinchon, Toulouse, France; Albert Stadler, Herrliberg, Switzerland; and the proposer.

An integral along a curve

1204. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $(x+iy)^n := P_n(x,y) + i\,Q_n(x,y)$ for $n = 1, 2, \ldots$, and let $f(x,y)$ be a continuous function along a (simple, closed, smooth) curve C in $\mathbb{R}^2$ such that the following equation holds:

$$\int_C f(x,y)dP_n(x,y) = \int_C f(x,y)dQ_n(x,y) = 0, n = 1, 2, \ldots.$$

Prove that $f(x,y)$ is constant along C.

Solution by Eugene Herman, Grinnell College, Grinnell, IA.

Concerning the curve C, we need only assume that it is smooth. We show that f is identically zero on C. By definition,

$$\int_C f(x,y)\,dP_n(x,y) = \int_C f(x,y)\left(\frac{\partial P_n}{\partial x}\,dx + \frac{\partial P_n}{\partial y}\,dy\right)$$

and similarly for the second integral. Thus, after multiplying the second integral by i and adding it to the first integral, we have

$$\begin{aligned}
0 &= \int_C f(x,y)\left(\left(\frac{\partial P_n}{\partial x} + i\frac{\partial Q_n}{\partial x}\right)dx + \left(\frac{\partial P_n}{\partial y} + i\frac{\partial Q_n}{\partial y}\right)dy\right)\\
&= \int_C f(x,y)\left(\frac{\partial}{\partial x}(x+iy)^n\,dx + \frac{\partial}{\partial y}(x+iy)^n\,dy\right)\\
&= n\int_C f(x,y)\left((x+iy)^{n-1}\,dx + i(x+iy)^{n-1}\,dy\right)
\end{aligned}$$

Let $z = x + iy$, and note that $f(x,y)$ can also be considered as a function of z, since $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2i$. Thus, denote $f(x,y)$ as $F(z)$. Therefore

$$\int_C F(z)z^{n-1}dz = 0, \quad n = 1, 2, \ldots$$

and so

$$\int_C F(z)p(z)\,dz = 0 \quad \text{for all complex polynomials } p$$

By the Weierstrass approximation theorem and the fact that C is compact, there exists a sequence $\langle p_n\rangle$ of complex polynomials that converges uniformly on C to F. Let $\alpha(t), a \le t \le b$, be a continuously differentiable parametrization of C with positive derivative. Then

$$0 = \int_C F(z)p_n(z)\,dz = \int_a^b F(\alpha(t))p_n(\alpha(t))\alpha'(t)\,dt$$

which converges to $\int_a^b (F(\alpha(t)))^2\alpha'(t)\,dt$ since F and α' are bounded. Since this integral is zero and $\alpha' > 0$, $F = 0$ on C and so $f = 0$ on C.

Also solved by the proposer.

A condition causing a prime ring to be commutative

1205. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let R be a (associative) prime ring, and suppose that $f\colon R \to R$ is a surjective ring endomorphism. If $[a, f(a)] = 0$ for all $a \in R$, prove that R is commutative (recall that the *commutator* $[xy] := xy - yx$ for $x, y \in R$, and R is *prime* if R is nontrivial and for all $a, b \in R$: if $arb = 0$ for all $r \in R$, then either $a = 0$ or $b = 0$).

Solution by the proposer, expanded by the editor.

Applying the condition $[f(a), a] = [a, f(a)]$ to $a + b$ and using linearity gives $[f(a), b] = [a, f(b)]$ for all $a, b \in R$. Hence,

$$\begin{aligned}[f(a)f(b), c] &= f(a)[f(b), c] + [f(a), c]f(b) = [ab, f(c)] = a[b, f(c)]\\ &= a[b, f(c)] + [a, f(c)]b,\end{aligned}$$

and we have

$$(f(a) - a)[b, f(c)] + [a, f(c)](f(b) - b) = 0 \text{ for all } a, b, c \in R.$$

Set $c = b$ and find that $[a, f(b)](f(b) - b) = 0$ for all $a, b \in R$. Then

$$0 = [ax, f(b)](f(b) - b) = [a, f(b)]x(f(b) - b).$$

The second equality can be seen as follows. By expanding, it is successively equivalent to

$$\begin{aligned} a\left(x(f(b)^2 - xf(b)b - f(b)xf(b) + f(b)xb\right) &= 0\\ a\left([x, f(b)]f(b) + [f(b), xb]\right) &= 0\\ a\left([f(x), b]f(b) + [b, f(xb)]\right) &= 0\\ a\left([f(x), b]f(b) + [b, f(x)]f(b)\right) &= 0\\ a\left([f(x), b] + [b, f(x)]\right) f(b) &= 0,\end{aligned}$$

which is true since $[b, f(x)] = -[f(x), b]$. Thus $[a, f(b)] = 0$ or $f(b) = b$ for all $a, b \in R$, because R is a prime ring.

If $w \in R \setminus Z(R)$, where $Z(R)$ denotes the center of R, then

$$[a, f(w)] = [f(a), w] \neq 0 \text{ for some } a \in R$$

(because the mapping f is onto), and it follows that $w = f(w)$. Additivity of the mapping f forces $f(a) = a$ for all $a \in R$, contrary to the assumption. Thus $R \setminus Z(R)$ is empty and R is commutative.

No other solutions were received.

Editor's note: Greg Dresden, the proposer of problem 1186, a solution to which appeared in the November 2020 issue, pointed out the following typos in the solution.

- At the bottom of page 389, the matrix on the far left of the last line has only three entries but should have four entries; this is repeated at the top of page 390.
- Near the top of page 390, the first fraction on the right after the line "This leads to the desired closed-form expression" has "$(3F - n + 1 + F_n)$" in the denominator, but that's supposed to be "$(3F_{n+1} + F_n)$."