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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, and Fuzhen Zhang.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by December 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12055. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let $a_1, a_2, \ldots$ be a sequence of nonnegative integers with $a_1 \geq a_2 \geq \cdots$ and with finite sum. For a positive integer j, let b_j be the number of indices i such that $a_i \geq j$. (The sequence $b_1, b_2, \ldots$ is the *conjugate* of $a_1, a_2, \ldots$.) Prove that the multisets $\{a_1 + 1, a_2 + 2, \ldots\}$ and $\{b_1 + 1, b_2 + 2, \ldots\}$ are equal. For example, if $\langle a_i \rangle = \langle 5, 3, 2, 2, 0, 0, 0, \ldots \rangle$, then $\langle b_j \rangle = \langle 4, 4, 2, 1, 1, 0, 0, \ldots \rangle$, and the corresponding multisets are $\{6, 5, 5, 6, 5, 6, 7, 8, \ldots\}$ and $\{5, 6, 5, 5, 6, 6, 7, 8, \ldots\}$.

12056. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania, Kadir Altintas, Emirdağ, Turkey, and Florin Stanescu, Gaesti, Romania.* Let $ABCD$ be a rectangle inscribed in a circle S of radius R, and let P be a point inside S. The lines AP, BP, CP, and DP intersect S a second time at K, L, M, and N, respectively. Prove $AK^2 + BL^2 + CM^2 + DN^2 \geq 16R^4/(R^2 + OP^2)$.

12057. *Proposed by Peter Kórus, University of Szeged, Szeged, Hungary.*
(a) Calculate the limit of the sequence defined by $a_1 = 1$, $a_2 = 2$, and

$$a_{2k+1} = \frac{a_{2k-1} + a_{2k}}{2} \quad \text{and} \quad a_{2k+2} = \sqrt{a_{2k}\, a_{2k+1}}$$

for positive integers k.

(b) Calculate the limit of the sequence defined by $b_1 = 1$, $b_2 = 2$, and

$$b_{2k+1} = \frac{b_{2k-1} + b_{2k}}{2} \quad \text{and} \quad b_{2k+2} = \frac{2\, b_{2k}\, b_{2k+1}}{b_{2k} + b_{2k+1}}$$

for positive integers k.

12058. *Proposed by Max A. Alekseyev, George Washington University, Washington, DC.* Let b be an integer greater than 1. For a positive integer n, let $u_b(n)$ be the number of nonzero digits in the base b representation of n. Prove that for any positive integers n and k, there exists a positive integer m such that $u_b(mn) = u_b(n) + k$.

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12059. *Proposed by George Stoica, Saint John, NB, Canada.* Let n be an integer greater than 1, and let R be the ring of polynomials in the variables $x_1, \dots, x_n$ with real coefficients. Let S be the ideal in R generated by the elementary symmetric polynomials $e_1, \dots, e_n$, where

$$e_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

for $1 \le k \le n$. The *degree* of a monomial $x_1^{m_1} \cdots x_n^{m_n}$ is $m_1 + \cdots + m_n$. Prove that $n(n-1)/2$ is the largest degree among all monomials that do not belong to S.

12060. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $\zeta(3)$ be Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$, and let H_n be the nth harmonic number $1 + 1/2 + \cdots + 1/n$. Prove

$$\sum_{n=2}^{\infty} \frac{H_n H_{n+1}}{n^3 - n} = \frac{5}{2} - \frac{\pi^2}{24} - \zeta(3).$$

12061. *Proposed by Dao Thanh Oai, Thai Binh, Viet Nam, and Le Viet An, Hue, Viet Nam.* Two triangles ABC and $A'B'C'$ in the plane are *perspective from a point* if the lines AA', BB', and CC' are concurrent (the common point is the *perspector*) and are *perspective from a line* if the points of intersection of AB and $A'B'$, of AC and $A'C'$, and of BC and $B'C'$ are collinear (the common line is the *perspectrix*). Desargues's theorem states that two triangles are perspective from a point if and only if they are perspective from a line. Consider three triangles, each pair of which are perspective from a point, hence per Desargues's theorem perspective from a line. Show that the three perspectrices are identical if and only if the three perspectors are collinear.

SOLUTIONS

A Triangle out of Pieces

11934 [2016, 832]. *Proposed by Leonard Giugiuc, Drobotu Turnu Severin, Romania.* Let ABC be an isosceles triangle, with $|AB| = |AC|$. Let D and E be two points on side BC such that $D \in BE$, $E \in DC$, and $m(\angle DAE) = \frac{1}{2} m(\angle A)$. Describe how to construct a triangle XYZ such that $|XY| = |BD|$, $|YZ| = |DE|$, and $|ZX| = |EC|$. Also, compute $m(\angle BAC) + m(\angle YXZ)$ (in radians).

Solution by Pál Péter Dályay, Szeged, Hungary. Write α, β, γ for the radian measures of the angles at A, B, C, respectively. Construct three circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with center A and radii r_1, r_2, r_3, respectively, with $r_1 = |AB|$, $r_2 = |AD|$, $r_3 = |AE|$. Let X be the intersection of the ray from A to the midpoint of BC with $\mathcal{C}_1$, let Y be the intersection of ray AE with $\mathcal{C}_2$, and let Z be the intersection of the ray AD with $\mathcal{C}_3$. We claim that $\triangle XYZ$ meets the required conditions.

Let $\triangle ABD$ be rotated around A by $\alpha/2$ to bring B to X and D to Y. Since $\triangle ABD$ is congruent to $\triangle AXY$, we have $|XY| = |BD|$ and $m(\angle AXY) = m(\angle ABD) = \beta$.

Similarly, let $\triangle ACE$ be rotated around A by $\alpha/2$ to bring C to X and E to Z. As before we conclude $|ZX| = |EC|$ and $m(\angle AXZ) = m(\angle ACE) = \gamma$.

Triangles ADE and AYZ are congruent, since they share an angle A, $|AY| = |AD|$, and $|AZ| = |AE|$. Thus $|YZ| = |DE|$, and triangle XYZ satisfies the required conditions.

Since $m(\angle YXZ) = m(\angle AXY) + m(\angle AXZ) = \beta + \gamma$, we have $m(\angle BAC) + m(\angle YXZ) = \alpha + (\beta + \gamma) = \pi$.

Editorial comment. The problem as originally published had $\angle XYZ$ for the last angle where $\angle YXZ$ was intended.

Also solved by R. B. Campos (Spain), R. Chapman (U. K.), I. Dimitric, J. Han (Korea), E. Ionascu, B. Karaivanov (U. S. A) & T. S. Vassilev (Canada), O. Kouba (Syria), O. P. Lossers (Netherlands), M. Meyerson, R. Stong, Armstrong State University Problem Solvers, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, and the proposer.

Hidden Mersenne

11936 [2016, 941]. *Proposed by William Weakley, Indiana University–Purdue University at Fort Wayne, Fort Wayne, IN.* Let S be the set of integers n such that there exist integers i, j, k, m, p with $i, j \geq 0$, $m, k \geq 2$, and p prime, such that $n = m^k = p^i + p^j$.
(a) Characterize S.
(b) For which elements of S are there two choices of (p, i, j)?

Solution by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND.
(**a**) The set S is the union of three sets: (1) $\{2^d : d \geq 2\}$, (2) $\{(2^t 3)^2 : t \geq 0\}$, and (3) $\{(2p^t)^k : t \geq 0 \text{ and } p = 2^k - 1 \text{ (a Mersenne prime)}\}$.

First, we prove that if $1 + p^d = v^k$, where p is a prime, $d \geq 1$, and $v, k \geq 2$, then either $p = 2$ and $d = 3$ (that is, $1 + 2^3 = 3^2$), or $p = 2^k - 1$ (a Mersenne prime) and $d = 1$ (so $1 + (2^k - 1) = 2^k$).

We prove this claim in two cases. Suppose first that $p = 2$ and $1 + 2^d = v^k$. In this case, $2^d = v^k - 1 = (v - 1)(v^{k-1} + \cdots + v + 1)$. Since all factors of 2^d are even, it follows that v is odd and $v^{k-1} + \cdots + v + 1$ is even, so k must be even, with $k = 2t$ for some $t \in \mathbb{N}$. This yields $2^d = v^k - 1 = (v^t - 1)(v^t + 1)$, so $v^t - 1$ and $v^t + 1$ are powers of 2 differing by 2. Thus they must be 2 and 4, so $v^t - 1 = 2$, and this implies $v = 3$, $t = 1$, and $d = 3$.

In the remaining case, p is an odd prime. Factor k as $p^t m$, where $t \geq 0$ and $p \nmid m$, and let $w = v^{p^t}$. We have $p^d = v^k - 1 = w^m - 1 = (w - 1)(w^{m-1} + \cdots + w + 1)$. If $w - 1 > 1$, then p divides both $w - 1$ and $w^{m-1} + \cdots + w + 1$, but then $w \equiv 1 \bmod p$ and $w^{m-1} + \cdots + w + 1 \equiv m \bmod p$. Hence p divides m, a contradiction. Therefore $w = 2$, and so $v = 2$ and $1 + p^d = 2^k$. Since we are given $k \geq 2$, it follows that $1 + p^d \equiv 0 \bmod 4$, so d is odd. If $d > 1$, then $d = qs$, where q is an odd prime and s is odd. We must have $s = 1$, since otherwise

$$2^k = p^d + 1 = (p^q + 1)\big((p^q)^{s-1} + (p^q)^{s-2} - \cdots - p^q + 1\big),$$

and the second factor is an odd number larger than 1. Thus d is an odd prime, and p has order $2d$ modulo 2^k, because $p^2 \not\equiv 1 \bmod 2^k$ (since $1 < p^2 < p^d < 2^k$) and $p^d \equiv -1 \bmod 2^k$. Thus $2d$ divides $\phi(2^k) = 2^{k-1}$, a contradiction. We conclude $d = 1$, and $p = 2^k - 1$ must be a Mersenne prime, finishing the proof of the claim.

Now consider the general case $n = m^k = p^i + p^j$. If $i = j$, then $m^k = 2p^i$, so $p = 2$ and $n = 2^{i+1}$. Thus n can be 2^d with $d \geq 2$ ($d = 1$ is excluded by $m, k \geq 2$). If $i < j$, then $m^k = p^i(1 + p^{j-i})$. Since p^i and $1 + p^{j-i}$ are relatively prime, we have $i = kt$ for some $t \geq 0$, and $1 + p^{j-i} = v^k$ for some $v \geq 2$. By our claim we have either $p = 2$ with $j - i = 3$ (so $v = 3$ and $k = 2$), or $p = 2^k - 1$ is a Mersenne prime with $j - i = 1$ (so $v = 2$). Thus $n = (2^t 3)^2 = 2^{2t} + 2^{2t+3}$ for some $t \geq 0$, or $n = (2p^t)^k = p^{kt} + p^{kt+1}$ for some $t \geq 0$, where $p = 2^k - 1$ is a Mersenne prime. Hence the set S is as claimed above.
(**b**) We further assume $i \leq j$ to exclude two such trivial representations obtained by switching i and j, so each member of (1), (2), or (3) has only one representation in that family.

Clearly, values of n in (1) and (2) cannot be the same. If n is in both (1) and (3), then $t = 0$ and $d = k$ (so $n = 2^k$, where $2^k - 1$ is a Mersenne prime), while if n has a representation in (2) and (3), then $p = 3$ (which is a Mersenne prime), $t = 1$ (in both representations), and $k = 2$ (so $n = 36$). Hence the only numbers in S with two different representations are 36 (represented as $2^2 + 2^5$ and $3^2 + 3^3$) and 2^k (represented as $2^{k-1} + 2^{k-1}$ and $(2^k - 1)^0 + (2^k - 1)^1$) whenever $2^k - 1$ is a Mersenne prime.

Editorial comment. To simplify the proof, several solvers referred to Catalan's conjecture (proved by Mihăilescu in 2004) that the only consecutive integers that are powers of integers with exponents at least 2 are 2^3 and 3^2.

Also solved by B. Karaivanov (U. S. A) & T. S. Vassilev (Canada), GCHQ Problem Solving Group (U. K.), and NSA Problems Group. Part (**a**) also solved by Y. J. Ionin, M. Josephy (Costa Rica), O. P. Lossers (Netherlands), R. Stong, and the proposer.

A Double Integral for the Digamma Function

11937 [2016, 941]. *Proposed by Juan Carlos Sampedro, University of the Basque Country, Leioa, Spain.* Let s be a complex number that is not a zero of the gamma function $\Gamma(s)$. Prove

$$\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y}{(1 - xy)\log(xy)}\, dx\, dy = \frac{\Gamma'(s)}{\Gamma(s)}.$$

Composite solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands, and Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. No finite complex number is a zero of $\Gamma(s)$, but we must assume $\operatorname{Re} s > 0$ for the integral to converge. Write the integral as

$$I(s) = -\int_0^1 \int_0^1 \frac{1 - (xy)^{s-1}}{1 - xy} \frac{dx\, dy}{\log(xy)} + \int_0^1 \int_0^1 \frac{1 - y}{1 - xy} \frac{dx\, dy}{\log(xy)}.$$

Notice that $(1 - (xy)^{s-1})/(1 - xy)$ has finite limit as $xy \to 1$, the functions x^{s-1} and y^{s-1} are integrable at 0, and $\int_0^1 \int_0^1 dx\, dy/|\log(xy)| < +\infty$. Therefore, the first integral converges absolutely. Since $0 \le (1 - y)/(1 - xy) \le 1$ whenever $0 < x, y < 1$, the second integral converges absolutely as well.

Now $I(s)$ is an analytic function of s in the right half-plane, so it suffices to prove the result for $0 < s < 1$. In this case, the integrand is real and has constant sign, so we may interchange the order of integration. Thus,

$$\begin{aligned} I(s) &= \int_0^1 \left(\int_0^1 \frac{(xy)^{s-1} - y}{1 - xy} \frac{dx}{\log(xy)} \right) dy = \int_0^1 \left(\int_0^y \frac{t^{s-1} - y}{y(1 - t)\log t}\, dt \right) dy \\ &= \int_0^1 \frac{1}{(1 - t)\log t} \left(\int_t^1 \frac{t^{s-1} - y}{y}\, dy \right) dt \\ &= \int_0^1 \frac{-t^{s-1}\log t - (1 - t)}{(1 - t)\log t}\, dt = \int_0^1 \left(\frac{-t^{s-1}}{1 - t} - \frac{1}{\log t} \right) dt. \end{aligned}$$

This is a well-known integral representation of the digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$ due to Gauss.

Also solved by M. Arnold, A. Berkane (Algeria), P. Bracken, R. Chapman (U. K.), H. Chen, B. Davis, C. Georghiou (Greece), G. Greubel, J.-P. Grivaux (France), J. A. Grzesik, E. Herman, R. Nandan, M. O'Brien, M. Omarjee (France), F. Perdomo & Á. Plaza (Spain), P. Perfetti (Italy), S. Sharma, A. Stadler (Switzerland),

R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), M. Wildon (U. K.), Y. Zhang, GCHQ Problem Solving Group (U. K.), and the proposer.

An Inequality for Triangles

11938 [2016, 941]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, Macedonia.* Let a, b, c be the lengths of the sides of a triangle, and let A be its area. Let R and r be the circumradius and inradius of the triangle, respectively. Prove

$$a^2+b^2+c^2 \geq (a-b)^2+(b-c)^2+(c-a)^2+4A\sqrt{3+\frac{R-2r}{R}}.$$

Solution by John G. Heuver, Grande Prairie, AB, Canada. Let $\angle A=\alpha$, $\angle B=\beta$, and $\angle C=\gamma$. By the law of cosines

$$a^2=b^2+c^2-2bc\cos\alpha=(b-c)^2+2bc(1-\cos\alpha)=(b-c)^2+4A\tan\frac{\alpha}{2},$$

where we have used $2A=bc\sin\alpha$ and $(1-\cos\alpha)/\sin\alpha=\tan(\alpha/2)$. It follows that

$$a^2+b^2+c^2=(a-b)^2+(b-c)^2+(c-a)^2+4A\left(\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right).$$

We have

$$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\frac{4R+r}{s},$$

where s is the semiperimeter of the triangle. (This is equation 83 on page 59 of D. S. Mitrinovic (1989), *Recent Advances in Geometric Inequalities*, Dordrecht: Kluwer.) Kooi's inequality

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$$

(see, for example, item 5.7 in O. Bottema, et. al. (1969), *Geometric Inequalities,* Groningen: Wolters-Noordhoff) then gives

$$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2} \geq \sqrt{3+\frac{R-2r}{R}}.$$

This completes the proof. Equality holds if and only if the triangle is equilateral.

Also solved by A. Ali (India), R. Boukharfane (France), P. P. Dályay (Hungary), L. Giugiuc (Romania), B. Karaivanov (U. S. A.) and T. S. Vassilev (Canada), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, D. Moore, R. Nandan, P. Nüesch (Switzerland), P. Perfetti (Italy), V. Schindler (Germany), M. Stănean (Romania), R. Stong, M. Vowe (Switzerland), T. Wiandt, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Summing Errors in Approximations to Euler's Constant

11939 [2016, 941]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Find

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\log(k)-\gamma-\frac{1}{2k}+\frac{1}{12k^2}\right).$$

Here γ is Euler's constant.

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. Let $H_k = 1 + 1/2 + \cdots + 1/k$. We have

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} = \sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n} 1 = \sum_{j=1}^{n}\frac{n+1-j}{j} = (n+1)H_n - n.$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{n}\left(H_k - \log(k) - \gamma - \frac{1}{2k}\right) &= (n+1)H_n - n - \log(n!) - n\gamma - \frac{H_n}{2} \\
&= \left(n+\frac{1}{2}\right)\left(\log(n) + \gamma + \frac{1}{2n} + O(1/n^2)\right) - n - n\gamma \\
&\quad - \left(n\log(n) - n + \frac{\log(2\pi)}{2} + \frac{\log(n)}{2} + O(1/n)\right) \\
&= \frac{1+\gamma-\log(2\pi)}{2} + O(1/n),
\end{aligned}$$

where we have used the approximations $H_n = \log(n) + \gamma + \frac{1}{2n} + O(1/n^2)$ and $\log(n!) = n\log(n) - n + \frac{\log(2\pi)}{2} + \frac{\log(n)}{2} + O(1/n)$. Also,

$$\sum_{k=1}^{\infty}\frac{1}{12k^2} = \frac{1}{12}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{72}.$$

Combining these results, we obtain

$$\sum_{k=1}^{\infty}\left(H_k - \log(k) - \gamma - \frac{1}{2k} + \frac{1}{12k^2}\right) = \frac{1+\gamma-\log(2\pi)}{2} + \frac{\pi^2}{72}.$$

Editorial comment. Several solvers noted that the requested sum, without the final term $1/(12k^2)$, appears as Problem 3.42 on page 195 of O. Furdui (2013), *Limits, Series, and Fractional Part Integrals: Problems in Mathematical Analysis*, New York: Springer. The more general formula

$$\sum_{k=1}^{\infty}\left(H_{pk} - \log(pk) - \gamma - \frac{1}{2pk}\right) = \frac{\log(p)+\gamma-\log(2\pi)}{2} + \frac{1}{2p} + \frac{\pi}{2p^2}\sum_{k=1}^{p-1} k\cot\left(\frac{k\pi}{p}\right),$$

where p is a positive integer, appears in O. Kouba (2016), Inequalities for finite trigonometric sums. An interplay: with some series related to harmonic numbers, *J. Inequal. Appl.*, Paper No. 173, 15 pp.

Also solved by A. Balfaqih (Yemen), A. Berkane (Algeria), R Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, R. Guculiére (France), R. Dutta (India), O. Furdui (Romania), N. Ghosh, M. L. Glasser, J. A. Grzesik, L. Han, E. A. Herman, E. J. Ionaşcu, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), C. W. Lienhard, O. P. Lossers (Netherlands), G. N. Macris, P. Magli (Italy), C. R. McCarthy, R. Nandan, P. Perfetti (Italy), F. A. Rakhimjanovich (Uzbekistan), E. Schmeichel, A. Stadler (Switzerland), A. Stenger, R. Stong, M. Vowe (Switzerland), S. Wagon, H. Widmer (Switzerland), J. Zacharias, Y. Zhang, GCHQ Problem Solving Group (U. K.), and the proposer.

A Hypergeometric Identity

11940 [2016, 942]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $T_n = n(n+1)/2$ and $C(n,k) = (n-2k)\binom{n}{k}$. For $n \geq 1$, prove

$$\sum_{k=0}^{n-1} C(T_n, k)C(T_{n+1}, k) = \frac{n^3 - 2n^2 + 4n}{n+2}\binom{T_n}{n}\binom{T_{n+1}}{n}.$$

Solution I by Pierre Lalonde, Kingsey Falls, QC, Canada. Let m be a positive integer. We prove by induction on m the more general formula

$$\sum_{k=0}^{m-1} C(T_n, k)C(T_{n+1}, k) = \frac{m^2(n^2 + 2n - 4m + 4)}{n(n+2)}\binom{T_n}{m}\binom{T_{n+1}}{m}.$$

For $m = 1$ both sides give $T_n T_{n+1}$. Given the formula for m, we compute

$$\begin{aligned}
\sum_{k=0}^{m} C(T_n, k)C(T_{n+1}, k) &= \sum_{k=0}^{m-1} C(T_n, k)C(T_{n+1}, k) + C(T_n, m)C(T_{n+1}, m) \\
&= \left(\frac{m^2(n^2 + 2n - 4m + 4)}{n(n+2)} + (T_n - 2m)(T_{n+1} - 2m)\right)\binom{T_n}{m}\binom{T_{n+1}}{m} \\
&= \frac{(n^2 + 2n - 4m)}{n(n+2)}\frac{(n^2 + n - 2m)(n^2 + 3n - 2m + 2)}{4}\binom{T_n}{m}\binom{T_{n+1}}{m} \\
&= \frac{(m+1)^2(n^2 + 2n - 4m)}{n(n+2)}\frac{(T_n - m)(T_{n+1} - m)}{(m+1)^2}\binom{T_n}{m}\binom{T_{n+1}}{m} \\
&= \frac{(m+1)^2(n^2 + 2n - 4m)}{n(n+2)}\binom{T_n}{m+1}\binom{T_{n+1}}{m+1},
\end{aligned}$$

where the step from the second to the third line is easy (though tedious) to check. The special case $m = n$ gives the desired result.

Solution II by Akalu Tefera, Grand Valley State University, Allendale, MI. Dividing both sides of the desired equality by its right side yields $\sum_{k=0}^{n-1} F(n, k) = 1$, where

$$F(n, k) = \frac{n+2}{n^3 - 2n^2 + 4n}\frac{C(T_n, k)C(T_{n+1}, k)}{\binom{T_n}{n}\binom{T_{n+1}}{n}}.$$

Applying Gosper's algorithm to $F(n, k)$ produces a rational function

$$R(n, k) = \frac{4k^2(n^2 + 2n - 4k + 4)}{n(n+2)(n^2 + n - 4k)(n^2 + 3n - 4k + 2)}$$

such that setting $G(n, k) = F(n, k)R(n, k)$ yields $F(n, k) = G(n, k+1) - G(n, k)$, which can be confirmed easily. Summing both sides of this equality with respect to k then gives the telescoping sum

$$\sum_{k=0}^{n-1} F(n, k) = \sum_{k=0}^{n-1}\big(G(n, k+1) - G(n, k)\big) = G(n, n) - G(n, 0) = 1.$$

Also solved by R. Chapman (U. K.), R. Stong, R. Tauraso (Italy), and the proposer.

Rate of Convergence for an Integral

11941 [2016, 492]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let

$$L = \lim_{n\to\infty}\int_0^1 \sqrt[n]{x^n + (1-x)^n}\,dx.$$

(a) Find L.
(b) Find

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx - L \right).$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. (**a**) We prove $L = 3/4$. To see this, let $I_n = \int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx$. We have

$$\begin{aligned} I_n &= \int_0^{1/2} \sqrt[n]{x^n + (1-x)^n}\, dx + \int_{1/2}^1 \sqrt[n]{x^n + (1-x)^n}\, dx \\ &\geq \int_0^{1/2} (1-x)\, dx + \int_{1/2}^1 x\, dx = \frac{3}{4}. \end{aligned}$$

On the other hand, since $x \leq 1-x$ for $x \in [0, 1/2]$ and $1-x \leq x$ for $x \in [1/2, 1]$,

$$\begin{aligned} I_n &= \int_0^{1/2} \sqrt[n]{x^n + (1-x)^n}\, dx + \int_{1/2}^1 \sqrt[n]{x^n + (1-x)^n}\, dx \\ &\leq \int_0^{1/2} \sqrt[n]{2}\,(1-x)\, dx + \int_{1/2}^1 \sqrt[n]{2}\, x\, dx = \frac{3}{4}\sqrt[n]{2}. \end{aligned}$$

The squeeze theorem implies that $L = \lim_{n\to\infty} I_n = 3/4$.

(**b**) The limit is $\pi^2/48$. Notice that $\int_0^{1/2} (1-x) = \int_{1/2}^1 x\, dx = 3/8$. We claim

$$\lim_{n\to\infty} n^2 \left(\int_0^{1/2} \sqrt[n]{x^n + (1-x)^n}\, dx - \frac{3}{8} \right) = \frac{\pi^2}{96} \tag{1}$$

and

$$\lim_{n\to\infty} n^2 \left(\int_{1/2}^1 \sqrt[n]{x^n + (1-x)^n}\, dx - \frac{3}{8} \right) = \frac{\pi^2}{96}, \tag{2}$$

from which the required limit follows. To prove (1), we compute

$$\begin{aligned} &\lim_{n\to\infty} n^2 \left(\int_0^{1/2} \left(\sqrt[n]{x^n + (1-x)^n} - (1-x) \right) dx \right) \\ &= \lim_{n\to\infty} n^2 \left(\int_0^{1/2} (1-x) \left(\sqrt[n]{1 + \left(\frac{x}{1-x}\right)^n} - 1 \right) \right) dx \\ &= \lim_{n\to\infty} n^2 \left(\int_0^1 \frac{1}{(1+t)^3} \left(\sqrt[n]{1+t^n} - 1 \right) \right) dt \qquad (\text{letting } t = x/(1-x)) \\ &= \lim_{n\to\infty} n \left(\int_0^1 \frac{1}{(1+u^{1/n})^3} \left(\sqrt[n]{1+u} - 1 \right) u^{1/n-1} \right) du \qquad (\text{letting } u = t^n) \\ &= \int_0^1 \lim_{n\to\infty} \frac{1}{(1+u^{1/n})^3}\, n \left(\sqrt[n]{1+u} - 1 \right) u^{1/n-1}\, du \\ &= \frac{1}{8} \int_0^1 \frac{\ln(1+u)}{u}\, du = \frac{1}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{96}. \end{aligned}$$

Equation (2) follows from (1) upon substituting $1-x$ for x.

Editorial comment. Chen noted that the results can be generalized as follows. For part **(a)**: If f and g are nonnegative and integrable on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b \sqrt[n]{f(x)^n + g(x)^n}\, dx = \int_a^b \max\{f(x), g(x)\}\, dx.$$

For part **(b)**: If f is a positive continuous function on $[0, 1]$ with $f(0) = 1$ and $g(x)$ is continuous on $[0, 1]$, then

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{f(x^n)}g(x)\, dx - \int_0^1 g(x)\, dx \right) = g(1) \int_0^1 \frac{\ln f(x)}{x}\, dx.$$

Letting $f(x) = 1 + x$ and $g(x) = 1/(1 + x)^3$ yields the result in part **(b)**.

Also solved by R. Agnew, K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), B. E. Davis, R. Dutta (India), D. Fleischman, N. Ghosh, J.-P. Grivaux (France), L. Han, F. Holland (Ireland), E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), S. de Luxán (Germany) & Á. Plaza (Spain), M. Omarjee (France), N. Osipov (Russia), P. Perfetti (Italy), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

On Perpendicularity

11942 [2016, 492]. *Proposed by Florin Parvanescu, Slat, Romania.* In acute triangle ABC, let D be the foot of the altitude from A, let E be the foot of the perpendicular from D to AC, and let F be a point on segment DE. Prove that AF is perpendicular to BE if and only if $|FE|/|FD| = |BD|/|CD|$.

Solution by Wei-Kai Lai and John Risher (student), University of South Carolina Salkehatchie, Walterboro, SC. Note that since $\overrightarrow{AD} \cdot \overrightarrow{BD} = 0$,

$$\begin{aligned}
\overrightarrow{AF} \cdot \overrightarrow{BE} &= (\overrightarrow{AD} + \overrightarrow{DF}) \cdot (\overrightarrow{BD} + \overrightarrow{DE}) = \overrightarrow{AD} \cdot \overrightarrow{DE} + \overrightarrow{DF} \cdot \overrightarrow{BD} + \overrightarrow{DF} \cdot \overrightarrow{DE} \\
&= (\overrightarrow{AE} - \overrightarrow{DE}) \cdot \overrightarrow{DE} + |DF|\,|BD| \cos(\angle EDC) + |DF|\,|DE| \\
&= -|DE|^2 + |DF|\,|BD| \frac{|DE|}{|DC|} + |DF|\,|DE|. \qquad (1)
\end{aligned}$$

Consider first the necessity of the condition. When $AF \perp BE$, (1) yields $|DF|\,|BD| + |DF|\,|DC| = |DE|\,|DC|$. Since $|DE| = |DF| + |FE|$, we get

$$|DF|\,|BD| + |DF|\,|DC| = |DF|\,|DC| + |FE|\,|DC|,$$

which implies $|DF|\,|BD| = |FE|\,|DC|$ as required.

Now consider the sufficiency of the condition. Since $|DE| = |DF| + |FE|$, and $|FE|/|FD| = |BD|/|CD|$ is assumed, we can write (1) in the equivalent form

$$\begin{aligned}
\overrightarrow{AF} \cdot \overrightarrow{BE} &= -(|DF| + |FE|)^2 + |DF|\,|DE| \cdot \frac{|FE|}{|FD|} + |DF|(|DF| + |FE|) \\
&= -|DF|^2 - 2|DF| \cdot |FE| - |FE|^2 + (|DF| + |FE|)|FE| + |DF|^2 + |DF| \cdot |FE|.
\end{aligned}$$

This equals zero, and hence AF is perpendicular to BE, as desired.

Also solved by A. Ali (India), H. Bailey, R. Chapman (U. K.), P. P. Dályay (Hungary), P. De (India), I. Dimitrić, A. Fanchini, D. Fleischman, O. Geupel, L. Giugiuc (Romania), N. Grivaux (France), J. Han (South Korea), E. A. Herman, S. Hitotumatu (Japan), E. J. Ionaşcu, Y. Ionin, S.-H. Jeong (Korea), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, I. Mihăilă, J. Minkus, R. Nandan, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, T. Zvonaru & N. Stanciu (Romania), Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.