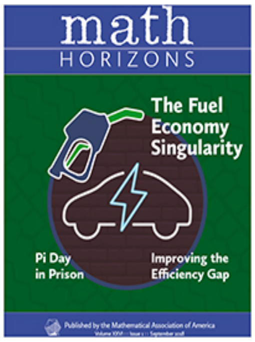

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The Playground

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THE PLAYGROUND

Welcome to the Playground.
Playground rules are posted on page 33, except for the most important one:
Have fun!

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Round Square (P374). In the plane, the ratio of the diagonal of a square to its side is always $\sqrt{2}$. Not so on the surface of a sphere. To be precise, say that four points A, B, C, and D on the surface of a sphere make a *spherical square* if the (minimal great-circle) distances between adjacent points (AB, BC, CD, and DA) are all equal and also the diagonals AC and BD are equal to each other, as illustrated in figure 1. Find the set of possible ratios $AC{:}AB$ and show that on a given sphere, this ratio uniquely determines the side length of the spherical square.

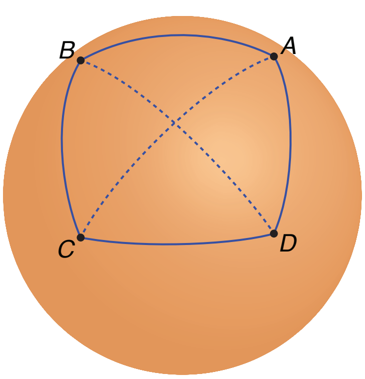

Figure 1. Spherical square *ABCD*.

Threefold Square (P375). A square is dissected into three similar, but pairwise noncongruent, rectangles. Determine the following difference: the ratio of the sides of the largest rectangle to the sides of the smallest rectangle, minus the corresponding ratio comparing the largest rectangle to the middle-sized rectangle.

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Burn Your Bridges (P376). This problem was submitted by Jim and Tom Wiseman, authors of "How Should Self-Driving Cars Drive?" on page 10. A simple model of drivers navigating roadways consists of a directed graph whose edges represent roads and vertices represent intersections. One node is labeled S (for "start") and another is labeled E (for "end"). An infinite sequence of drivers arrive in succession at node S, with *two* arriving in each time unit. Whenever a driver arrives at a node with more than one outgoing edge, the driver must choose along which edge to proceed. Each edge acts as a queue, with the one driver that entered that edge longest ago leaving the edge and arriving at the node to which the edge points, in each time unit. (Ties are broken by which driver arrived at S first.) Thus, entering a road that already has two cars on it will require three time units to traverse, rather than the single time unit for an empty road.

Furthermore, all drivers are assumed to be *Nash-optimizing*: Whenever faced with a choice, they choose an option that achieves the minimum time to reach E, based on the choices of all the earlier drivers (which it is assumed are known with perfect information). The *stable traversal time* of a sequence of drivers is the largest number of time units it takes infinitely many of the drivers to make their way from S to E. Unsurprisingly, if drivers are faced with multiple equally-good choices at some nodes, the choices they make can affect the stable traversal time.

For example, in the directed graph shown in figure 2, there is essentially only one way for the Nash-optimizing drivers to proceed: Whichever road the first driver arriving in a time unit chooses, the second driver chooses the other way to avoid the extra unit of time waiting in the queue. There are no other choices to make, so every car drives from S to E in the stable traversal time of four units.

What is the longest stable traversal time for Nash-optimizing drivers in the altered network of figure 3, with one additional road ("shortcut") added? For that traversal time, what is the smallest fraction of drivers that will use the added edge?

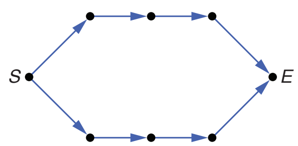

Figure 2. Two routes from *S* to *E*.

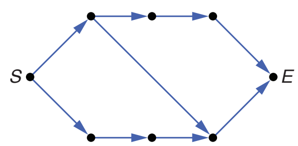

Figure 3. Network with shortcut added.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Base Tangent Inequality (P377). This traditional plane geometry problem devised by Yagub Aliyev of ADA University reaches us from Baku, Azerbaijan. Consider the acute triangle *ABC* with altitudes *AD* and *BE* intersecting at orthocenter *H*. Let *c* be the circle through *A*, *E*, and *H*. Suppose *DT* is tangent to *c* at *T* (see figure 4). Prove that $2\,|DT| \leq |BC|$, and determine under what conditions equality occurs.

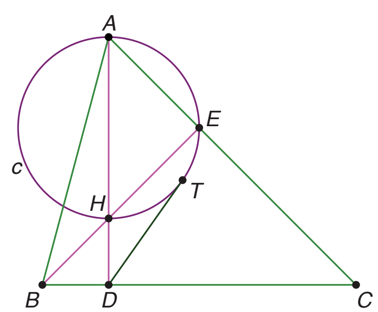

Figure 4. A tangent from the base of an altitude to the circle *c*.

FEBRUARY WRAP-UP

Problem 366. Place a circle of radius 1 so that it is doubly tangent to the parabola $y = x^2$ Then place an infinite sequence of additional circles, each tangent to the prior one and doubly tangent to the parabola (see figure 6). In **Circular Reasoning,** you were asked to find the probability that a point randomly selected from the interior of the parabola is inside one of the circles.

We received solutions from Brian Beasley (Presbyterian College), Alex Briasco-Brin (Freeport Middle School, Freeport, Maine), Dmitry Fleischman, Christopher Havens (Twin Rivers PMP), Randy Schwartz (Schoolcraft College), James Wittrig (Gustavus Adolphus College), Jessica Wycha (North Central College), and Problem Solving Teams from Georgia Southern University (the Armstrong Problem Solvers), Missouri State University, Mountain Lakes (New Jersey) High School, Seton Hall University (Karl Hendela, Leah Meissner, Robert Toth, and Shawn Weigel), and Skidmore College.

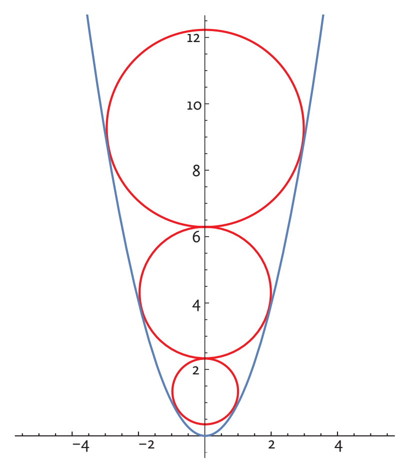

Figure 6. Circles inscribed in a parabola.

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Tantalizing Triangle (C22). Try to solve this gem, which Keith Enevoldsen billed as the "world's hardest easy geometry problem," with only classical synthetic geometry, no coordinates or trig. In isosceles triangle *ABC* with apex *C*, point *D* lies on *AC* and *E* lies on *BC*. Segment *BD* divides the angle at *B* into angles $\tau/6$ and $\angle CBD = \tau/18$. (Here, $\tau = 2\pi$ is the angle measure of a full circle.) Also, $\angle CAE = \tau/36$. Warning: Figure 5 is not to scale! Determine the measure of $\angle DEA$.

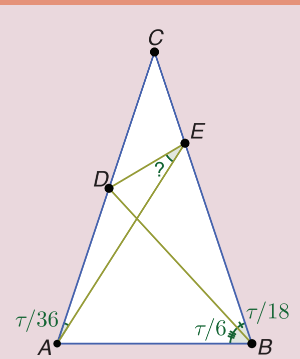

Figure 5. What is the unknown angle within isosceles triangle *ABC*?

The limiting probability is $\frac{\pi}{4}$. To see this, we first show that the center of the circle of radius 1 is the point $(0, \frac{5}{4})$. Suppose the center of the circle is the point $C = (0, h)$, and the circle is tangent to the parabola at the point $P = (a, a^2)$. Then the slope of the radius joining C and P is $(a^2 - h)/a$. On the other hand, this line is perpendicular to the tangent line because it's also tangent to the circle at P, so its slope is $-\frac{1}{2a}$. This gives $(a^2 - h)/a = -1/2a$, or $a^2 = h - \frac{1}{2}$. The distance from C to P is 1, so $(a^2 - h)^2 + a^2 = 1$. Replacing a^2 by $h - \frac{1}{2}$ in this expression gives $h = \frac{5}{4}$.

Continuing in this way, we find the radius of the second circle is 2, the third circle is 3, and so on. Further, the center of the circle of radius k is $(0, k^2 + \frac{1}{4})$. To finish the problem, we need to compare the area enclosed by the first n circles to the area of the region bounded above by the line $y = n^2 + n + \frac{1}{4}$ (this is the line tangent to the circle of radius n at its highest point on the y-axis) and bounded below by the parabola $y = x^2$. The total area enclosed by the circles is

$$\pi \sum_{k=1}^{n} k^2 = \pi \frac{2n^3 + 3n^2 + n}{6}.$$

On the other hand, the area of the region is $\frac{1}{6}(8n^3 + 12n^2 + 6n + 1)$. Then the ratio of these areas approaches $\frac{\pi}{4}$.

We point out that this limiting value matches the packing density of circles in the plane when the circles are centered at lattice points. It is also the same as the density we saw in problem 310 of the Playground in the September 2014 issue.

Problem 367. Given a positive rational number $\frac{a}{b}$ in lowest terms, you may perform the following operation on it as many times as you like: Add 1 to either the numerator or denominator, and then reduce the result to lowest terms. In **Fraction Action**, you were asked to show that it is possible to transform $\frac{a}{b}$ to any other positive rational $\frac{c}{d}$ (in lowest terms).

We received solutions from Dmitry Fleischman, Andrew Goetz (Armstrong Problem Solvers of Georgia State University), Abigail Kalina (Gustavus Adolphus College), Tim O'Neill (North Central College), James Swenson (University of Wisconsin-Platteville), Luke Szramowski (Slippery Rock University), the team of Nicholas Zelinsky and Elizabeth Newman (Seton Hall University), the Northwestern University Problem Solving Group, and the Skidmore College Problem Group. We also received two partial solutions from Taylor University students Jason Brahan, Caleb Fox, Jamie Netzley, Jakob Sprunger, Logan Tuckey, and Ariel Wentworth.

All the solutions we are aware of use Dirichlet's theorem on the distribution of prime numbers in arithmetic progressions. Here is one way to solve this problem. First, transform $\frac{a}{b}$ to 1. To do this, add 1 to the bottom repeatedly until the top and bottom have a common factor, then reduce the fraction. This step makes the top smaller. Continue in this fashion until all of the factors in the top have been canceled, so the fraction now has the form $\frac{1}{n}$ for some $n \geq 1$. (This process can be formalized as an induction on the size of the top.) If $n > 1$, the identical argument shows that now repeatedly adding 1 to the top will eventually reach $\frac{1}{1}$.

We now have 1. If $\frac{c}{d} < 1$, then add 1 to the bottom repeatedly until you get a prime number p with $p > c$ satisfying $p = k \cdot d - 1$ for

CAROUSEL SOLUTION

Because all the stipulated angles are multiples of $\tau/36$, the entire diagram (to scale) inscribes neatly into a regular 18-gon (in which the angle subtended by any edge from any vertex not on that edge is $\tau/36$). Figure 7 shows how nearly all the major lines in the diagram become diagonals of the 18-gon.

Reflect the entire diagram across the line MN connecting the midpoints of two opposite sides of the 18-gon. This configuration is shown in brown and orange in figure 7. Clearly, line $ED'DE'$ is perpendicular to MN and parallel to FF'. Moreover, line AEG is parallel to $F'H$. Therefore, $\angle DEA$ is the same as the exterior angle of an 18-gon, namely $\tau/18$.

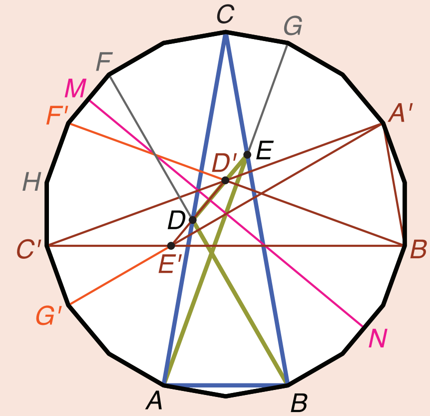

Figure 7. The reflected tantalizing triangle remains inscribed in the 18-gon.

some positive integer k. (The existence of such a prime is ensured by Dirichlet's theorem.) Finally, add 1 to the top repeatedly until the fraction looks like ck / p, then add 1 to the bottom one last time:

$$1 \to \frac{1}{2} \to \frac{1}{3} \to \cdots \to \frac{1}{p} \to \frac{2}{p} \to \cdots \to \frac{ck}{p} \to \frac{ck}{dk} = \frac{c}{d}.$$

Note that there can be no cancelling at any stage in getting to this point because p is prime. Finally, if $\frac{c}{d} > 1$, then reverse this procedure.

It would be interesting to see a solution that did not rely on Dirichlet's theorem.

Problem 368. Arthur Benjamin and Sam Miller's article "Challenging Knight's Tours" motivated this problem about a two-player game on an $m \times n$ chessboard. To start, player 1 places the mice on two different squares, then player 2 places the cheese on a third square. Then player 1 and player 2 alternate turns. Each turn, player 1 moves the mice one at a time one square horizontally or vertically (but not onto the other mouse). Player 2 may move the cheese in the same fashion or may pass. If a mouse eventually moves onto the square with the cheese, player 1 wins. Player 2 wins by perpetually avoiding that outcome. In **The Cheese Stands Alone,** you were asked to determine who has a winning strategy and what that strategy is.

We received solutions from Andrew Goetz (Armstrong Problem Solvers of Georgia State University), Catherine Way and Bianca Reilly (Seton Hall University), Tom Yuster (Lafayette College), and a partial solution from Jessica Doctor (Taylor University).

Player 1 can always win. We follow Tom Yuster's quick solution. Any time the cheese is in the yellow region (see figure 8), eat it. Anytime the cheese is in the orange region, move right. Otherwise move up or down to decrease the distance between the cheese and the mice. The result then follows by induction on the number of rows.

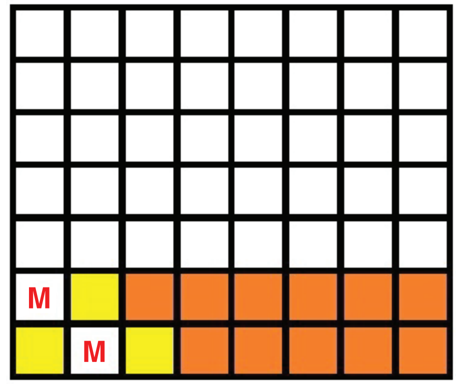

Figure 8. The mouse can always eat the cheese.

Problem 369. Define two sequences recursively as follows: $a_1 = b_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$ and $b_{n+1} = 3^{b_n}$. Then **Sequence Race** asked the following question: Is there a positive integer k satisfying $a_{n+k} > b_n$ for all $n > 2$?

We received solutions from Gabriel Augusto Correia (Universidade de Brasília, Brazil), Dmitry Fleischman, the Northwestern Problem Solving Group, and a partial solution from Zachery Huse, Zachary Saltzgaber, Timothy Hotchkiss, and Jordan Crawford (Taylor University).

The answer is yes; $k = 2$ works. The Northwestern group proved the stronger statement $a_{n+2} > (b_n)^2$ for $n \geq 1$. Here is their inductive argument.

First, we check the cases $n = 1, 2, 3$: $a_3 = 4 > 1 = b_1^2$, $a_4 = 16 > 9 = b_2^2$, and $a_5 = 65,536 > 729 = b_3^2$.

Next, assume $n \geq 3$, and, by induction, $a_{n+2} > (b_n)^2$. Then

$$\begin{aligned}(b_{n+1})^2 &= (3^{b_n})^2 = 3^{2b_n} = 9^{b_n} \\ &< (2^{b_3})^{b_n} \leq (2^{b_n})^{b_n} = 2^{b_n^2} < 2^{a_{n+2}} \\ &= a_{n+3}.\end{aligned}$$

Note from GG: It has been a pleasure editing this column for the past five years, and I wish to thank all of the people who sent me interesting solutions and problems. For more information about Christopher Havens, one of our contributors, please see the article on page 24.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems or solutions (including more elegant or extended solutions to Carousel problems) should be submitted to *MHproblems@maa.org* or *MHsolutions@maa.org*, respectively. Paper submissions may be sent to Glen Whitney, UCLA Math Dept., 520 Portola Plaza MS 6363, Los Angeles, CA 90095. Please include your name, email address, and school or institutional affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers. The deadline for submitting solutions to problems in this issue is November 9.

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