Math 114

Optional problems on Mathematical Induction

1. Suppose that $2n$ points are given in space, where $n \geq 2$. Altogether $n^2 + 1$ line segments are drawn between these points. Prove that there is at least one triangle (a set of three points which are joined pairwise by line segments).

2. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from other cars on its way around.

3. (2pts) Let n be any natural number. Consider all nonempty subsets of the set $\{1, 2, \ldots, n\}$, which do not contain any neighboring elements. Prove that the sum of the squares of the products of all numbers in these subsets is $(n+1)! - 1$. (For example, if $n = 3$, then such subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$, and $1^2 + 2^2 + 3^2 + (1 \cdot 3)^2 = 23 = 4! - 1$.)

4. Find the determinant of the $n \times n$ matrix A_n with entries

$$a_{ij} = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ if } |i - j| = 1 \\ 0 \text{ otherwise} \end{cases} .$$

   Hint: calculate the determinants of A_1, A_2, A_3, and A_4. Notice the pattern. Guess a formula for $\det A_n$, and then prove it by Mathematical Induction.