MATH 111

Test 3 (12/10/07) - Solutions

1. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Give an example of a relation from A to B that is not a function from A to B.

   The set $R = \{(1, 4)\}$ is a relation from A to B (because it is a subset of $A \times B$), but not a function from A to B (because e.g. the image of $2 \in A$ is undefined).

2. Let A be a set and $f : A \to A$ be onto. Prove that $f \circ f$ is onto.

   Let $y \in A$. Since f is onto, there exists $a \in A$ such that $f(a) = y$ and there exists $x \in A$ such that $f(x) = a$. Then $(f \circ f)(x) = f(f(x)) = f(a) = y$. Therefore $f \circ f$ is onto.

3. Determine whether $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^2 + 1}$ is one-to-one; onto; bijective.

   Since $f(-1) = \dfrac{1}{2} = f(1)$ and $-1 \neq 1$, f is not one-to-one. Since $\dfrac{1}{x^2 + 1} > 0$ for all $x \in \mathbb{R}$, $-1 \in \mathbb{R}$ is not in the image, so f is not onto. Finally, f is not bijective because it is not one-to-one and onto.

4. Let R be a relation on $\mathbb{Z}$ defined by $(a, b) \in R$ if and only if $2|(a+b)$. Determine whether R is an equivalence relation. If so, describe its distinct equivalence classes.

   We will prove that R is an equivalence relation.

   (1) For any $a \in \mathbb{Z}$, $2|(a + a)$, so $(a, a) \in R$. Thus R is reflexive.

   (2) If $(a, b) \in R$, then $2|(a + b)$. Then $2|(b + a)$, so $(b, a) \in R$. Thus R is symmetric.

   (3) If $(a, b) \in R$ and $(b, c) \in R$, then $2|(a + b)$ and $2|(b + c)$. Then $2|(a + b + b + c)$, i.e. $2|(a + 2b + c)$. Since $2|(2b)$, $2|(a + 2b + c - 2b)$, i.e. $2|(a + c)$. Therefore $(a, c) \in R$. Thus R is transitive.

   Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

   Equivalence classes are:

   $[0] = \{a \in \mathbb{Z} \mid (a, 0) \in R\} = \{a \in \mathbb{Z} \mid 2|a\} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$,

   $[1] = \{a \in \mathbb{Z} \mid (a, 1) \in R\} = \{a \in \mathbb{Z} \mid 2|(a + 1)\} = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$;

   since $[0] \cup [1] = \mathbb{Z}$, there are no other equivalence classes.

5. Prove that for any positive integer n,

   $$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n + 1)(n + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}.$$

   We will prove the identity by Mathematical Induction.

   Basis step: if $n = 1$, then $1 \cdot 2 \cdot 3 = \dfrac{1 \cdot 2 \cdot 3 \cdot 4}{4}$ is true.

Inductive step: assume that $1\cdot2\cdot3+2\cdot3\cdot4+\ldots+k(k+1)(k+2)=\dfrac{k(k+1)(k+2)(k+3)}{4}$ *for some* $k\in\mathbb{N}$. *We will prove that* $1\cdot2\cdot3+2\cdot3\cdot4+\ldots+(k+1)(k+2)(k+3)=\dfrac{(k+1)(k+2)(k+3)(k+4)}{4}$.

Observe that $1\cdot2\cdot3+2\cdot3\cdot4+\ldots+(k+1)(k+2)(k+3)=$

$(1\cdot2\cdot3+2\cdot3\cdot4+\ldots+k(k+1)(k+2))+(k+1)(k+2)(k+3)=$

$\dfrac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3)=$

$\dfrac{k(k+1)(k+2)(k+3)+4(k+1)(k+2)(k+3)}{4}=\dfrac{(k+1)(k+2)(k+3)(k+4)}{4}$.

6. (For extra credit) Let A be a set. Prove that if a function $f:A\to A$ is an equivalence relation on A, then it is bijective.

   If f *is an equivalence relation on* A, *then it is reflexive. Then for any* $a\in A$, $(a,a)\in f$, *i.e.* $f(a)=a$. *So* f *is the identity function on* A. *This function is one-to-one since* $f(a_1)=f(a_2)$ *implies* $a_1=a_2$ *and it is onto since for any* $b\in A$, *if we let* $a=b$, *then* $f(a)=b$. *Thus* f *is bijective.*