MATH 111
Test 2 (11/06/07) - Solutions

1. Let $a \in \mathbb{Z}$. Prove that if $3|a^2$, then $3|a$.

   Proof by contrapositive. Let $3 \not| a$. *Then either* $a = 3k+1$ *or* $a = 3k+2$ *for some* $k \in \mathbb{Z}$.

   Case I: $a = 3k+1$ *for some* $k \in \mathbb{Z}$. *Then* $a^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$. *Since* $3k^2 + 2k \in \mathbb{Z}$, $3 \not| a^2$.

   Case II: $a = 3k + 2$ *for some* $k \in \mathbb{Z}$. *Then* $a^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. *Since* $3k^2 + 4k + 1 \in \mathbb{Z}$, $3 \not| a^2$.

2. Prove that for any integer a, there exists an integer b such that $b > a$ and $a \equiv b \pmod 5$.

   For any $a \in \mathbb{Z}$, *let* $b = a + 5$. *Then* $b > a$. *Since* $0 \equiv 5 \pmod 5$, $a \equiv b \pmod 5$.

3. Prove or disprove. The equation $x^3 + 5x + 2 = 0$ has an integer solution.

   The statement is false. For any integer x, *either* $x \geq 0$ *or* $x \leq -1$. *We will consider these two cases.*

   Case I: $x \geq 0$. *Then* $x^3 + 5x + 2 \geq 0 + 0 + 2 = 2$, *so* $x^3 + 5x + 2 \neq 0$.

   Case II: $x \leq -1$. *Then* $x^3 + 5x + 2 \leq -1 - 5 + 2 = -4$, *so* $x^3 + 5x + 2 \neq 0$.

   In either case, x *is not a solution of the given equation.*

4. Prove or disprove. Let A and B be sets. Then $(A - B) \cup (A \cap B) = A$.

   The statement is true. First we will show that $(A - B) \cup (A \cap B) \subset A$. *Let* $x \in (A - B) \cup (A \cap B)$. *Then* $x \in (A - B)$ *or* $x \in (A \cap B)$. *In either case,* $x \in A$.

   Next we will show that $A \subset (A - B) \cup (A \cap B)$. *Let* $x \in A$. *If* $x \in B$, *then* $x \in (A \cap B)$. *If* $x \notin B$, *then* $x \in (A - B)$. *In either case,* $x \in (A - B) \cup (A \cap B)$.

5. Prove or disprove. There exists a largest rational number, i.e. a rational number a such that for any rational number b, $a \geq b$.

   The statement is false. We will prove this by contradiction. Suppose a *is a largest rational number. Consider* $b = a + 1$. *Then* $b > a$. *Also, since* a *is rational,* $a = \frac{m}{n}$ *for some* $m, n \in \mathbb{Z}$, $n \neq 0$. *Then* $b = a + 1 = \frac{m}{n} + 1 = \frac{m+n}{n}$. *Since* $m + n, n \in \mathbb{Z}$ *and* $n \neq 0$, b *is rational. So* b *is a rational number larger than* a. *Contradiction.*

6. (For extra credit) Prove or disprove. For any rational numbers a and b such that $a < b$, there exists an irrational number x such that $a < x < b$.

   The statement is true. For any rational numbers a *and* b, *consider* $x = a + \frac{b-a}{\sqrt{2}}$. *Since* $b - a > 0$, $x > a$. *Also, since* $\sqrt{2} > 1$, $x < a + (b - a) = b$. *Finally, we will prove that* x *is irrational. We will prove this by contradiction. Assume* x *is rational. Then* $a = \frac{k}{l}$, $b = \frac{m}{n}$, *and* $x = \frac{p}{q}$ *for some* $k, l, m, n, p, q \in \mathbb{Z}$, $l \neq 0$, $n \neq 0$, $q \neq 0$. *Thus* $\frac{p}{q} = \frac{k}{l} + \frac{\frac{m}{n} - \frac{k}{l}}{\sqrt{2}}$. *Equivalently,* $\frac{p}{q} - \frac{k}{l} = \frac{\frac{ml-kn}{nl}}{\sqrt{2}}$, *or* $\frac{pl-kq}{ql} = \frac{ml-kn}{\sqrt{2}nl}$. *Therefore* $\sqrt{2} = \frac{q(ml-kn)}{n(pl-kq)}$. *Since* $q(ml - kn), n(pl - kq) \in \mathbb{Z}$ *and* $n(pl - kq) \neq 0$ ($pl - kq \neq 0$ *because* $x - a \neq 0$), $\sqrt{2}$ *is rational. Contradiction.*