Fundamental Logical Equivalences

The following equivalences can be verified by constructing the corresponding truth tables.

Remark. The list below contains the most well-known and most often used equivalences. Some other operations share the same properties, some of which were discussed in class but omitted here. Don't memorize all those discussed in class, but the ones below are good to know.

T stands for True (or any compound statement that is always true), and F stands for False (or any compound statement that is always false).

1. Commutative laws:
$$P \vee Q \equiv Q \vee P$$
$$P \wedge Q \equiv Q \wedge P$$

2. Associative laws:
$$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$$
$$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$$

3. Distributive laws:
$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$
$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

4. Idempotent laws:
$$P \vee P \equiv P$$
$$P \wedge P \equiv P$$

5. Identity laws:
$$P \vee F \equiv P$$
$$P \wedge T \equiv P$$

6. Inverse laws:
$$P \vee \neg P \equiv T$$
$$P \wedge \neg P \equiv F$$

7. Domination laws:

$$P \vee T \equiv T$$

$$P \wedge F \equiv F$$

8. Absorption laws:

$$P \vee (P \wedge Q) \equiv P$$

$$P \wedge (P \vee Q) \equiv P$$

9. Double negation law:

$$\neg(\neg P) \equiv P$$

10. DeMorgan's laws:

$$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$

$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$$

11. Implication identity:

$$P \to Q \equiv (\neg P) \vee Q$$

12. Contrapositive identity:

$$P \to Q \equiv (\neg Q) \to (\neg P)$$

13. Biconditional identities:

$$P \leftrightarrow Q \equiv (P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$$

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$$

Tautologies and contradictions.

Definition. A compound statement (or a formula) is called a tautology if it is true for all combinations of truth values of all the propositional variables that appear in it.

Examples. $P \vee \neg P$ and $(P \wedge (P \to Q)) \to Q$ are tautologies. This can be easily verified using truth tables.

Definition. A compound statement (or a formula) is called a contradiction if it is false for all combinations of truth values of all the propositional variables that appear in it.

Examples. $P \wedge \neg P$ is a contradiction. This can be easily verified using a truth table.

Remark. If a compound statement A is a tautology, then $\neg A$ is a contradiction. Conversely, if A is a contradiction, then $\neg A$ is a tautology.

Relationship between logical equivalences, tautologies, and contradictions.

Suppose that A and B are (compound) statements (or formulas). Then the following are equivalent: $A \equiv B$, $A \leftrightarrow B$ is a tautology, and $A \oplus B$ is a contradiction. This directly follows from the definitions of these concepts.

Proving other equivalences and simplifying compound statements using the above fundamental equivalences.

Example 1. Prove that $\neg(P \to Q) \equiv P \wedge (\neg Q)$.

Proof:

$$\begin{aligned}
\neg(P \to Q) &\equiv \neg((\neg P) \vee Q) \qquad \text{(Implication identity)} \\
&\equiv (\neg(\neg P)) \wedge (\neg Q) \qquad \text{(DeMorgan's law)} \\
&\equiv P \wedge (\neg Q) \qquad \text{(Double negation law)}
\end{aligned}$$

Example 2. Prove that $\neg(P \leftrightarrow Q) \equiv (P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$.

Proof:

$$\begin{aligned}
\neg(P \leftrightarrow Q) &\equiv \neg((P \to Q) \wedge (Q \to P)) \qquad \text{(Biconditional identity)} \\
&\equiv (\neg(P \to Q)) \vee (\neg(Q \to P)) \qquad \text{(DeMorgan's law)} \\
&\equiv (P \wedge (\neg Q)) \vee (Q \wedge (\neg P)) \qquad \text{(by Example 1)} \\
&\equiv (P \wedge (\neg Q)) \vee ((\neg P) \wedge Q) \qquad \text{(Commutative law)}
\end{aligned}$$

Example 3. Simplify: $(Q \to P) \wedge ((\neg Q) \leftrightarrow (P \vee Q))$.

Solution: First rewrite $\to$ and $\leftrightarrow$ (and $\oplus$ when it's present in the expression) in terms of the other three operations, then use distributivity and see if any terms can be simplified.

$$\begin{aligned}
(Q \to P) \wedge ((\neg Q) \leftrightarrow (P \vee Q)) &\equiv ((\neg Q) \vee P) \wedge ((\neg Q) \leftrightarrow (P \vee Q)) \\
&\qquad \text{(Implication identity)} \\
&\equiv ((\neg Q) \vee P) \wedge \left[((\neg Q) \wedge (P \vee Q)) \vee ((\neg\neg Q) \wedge \neg(P \vee Q))\right] \\
&\qquad \text{(Biconditional identity)} \\
&\equiv ((\neg Q) \vee P) \wedge \left[((\neg Q) \wedge (P \vee Q)) \vee (Q \wedge \neg(P \vee Q))\right]
\end{aligned}$$

(Double negation law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge (P \vee Q)) \vee (Q \wedge ((\neg P) \wedge (\neg Q)))\big]$

(DeMorgan's law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge (P \vee Q)) \vee (Q \wedge ((\neg Q) \wedge (\neg P)))\big]$

(Commutative law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge (P \vee Q)) \vee ((Q \wedge (\neg Q)) \wedge (\neg P))\big]$

(Associative law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge (P \vee Q)) \vee (F \wedge (\neg P))\big]$ (Inverse law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge (P \vee Q)) \vee F\big]$ (Domination law)

$\equiv ((\neg Q) \vee P) \wedge \big[(\neg Q) \wedge (P \vee Q)\big]$ (Identity law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge P) \vee ((\neg Q) \wedge Q)\big]$ (Disrtibutive law)

$\equiv ((\neg Q) \vee P) \wedge \big[((\neg Q) \wedge P) \vee F\big]$ (Inverse law)

$\equiv ((\neg Q) \vee P) \wedge ((\neg Q) \wedge P)$ (Identity law)

$\equiv ((\neg Q) \wedge ((\neg Q) \wedge P)) \vee (P \wedge ((\neg Q) \wedge P))$ (Distributive law)

$\equiv ((\neg Q) \wedge ((\neg Q) \wedge P)) \vee (P \wedge (P \wedge (\neg Q)))$ (Commutative law)

$\equiv (((\neg Q) \wedge (\neg Q)) \wedge P) \vee ((P \wedge P) \wedge (\neg Q))$ (Associative law)

$\equiv ((\neg Q) \wedge P) \vee (P \wedge (\neg Q))$ (Idempotent law)

$\equiv (P \wedge (\neg Q)) \vee (P \wedge (\neg Q))$ (Commutative law)

$\equiv P \wedge (\neg Q)$ (Idempotent law)