MATH 110 **Lecture notes – 14**

Axioms

An axiom system consists of a set of formulas (called axioms) and some rules (called rules of inference). We say that an axiom system is **sound** if every formula that is derivable from this axiom system is valid (i.e. is a tautology). An axiom system is **complete** if every valid formula (i.e. every tautology) can be derived from this axiom system.

The following are two sound and complete axiom systems for the classical propositional logic. There are many other sound and complete axiom systems.

Axiom system 1:

(1) $X \to (Y \to X)$

(2) $(X \to (Y \to Z)) \to ((X \to Y) \to (X \to Z))$

(3) $X \wedge Y \to X$

(4) $X \wedge Y \to Y$

(5) $X \to (Y \to (X \wedge Y))$

(6) $X \to X \vee Y$

(7) $Y \to X \vee Y$

(8) $(X \to Z) \to ((Y \to Z) \to (X \vee Y \to Z))$

(9) $(X \to Y) \to ((X \to \neg Y) \to \neg X)$

(10) $X \to (\neg X \to Y)$

(11) $X \vee \neg X$

(12) $(X \leftrightarrow Y) \to (X \to Y)$

(13) $(X \leftrightarrow Y) \to (Y \to X)$

(14) $(X \to Y) \to ((Y \to X) \to (X \leftrightarrow Y))$

Rule of inference:

- (Modus Ponens) $\dfrac{X,\ X \to Y}{Y}$

The rule of Modus Ponens means that if X and $X \to Y$ are derivable from the axiom system, then so is Y.

Axiom system 2:

(1) $(X \wedge Y) \to X$

(2) $(X \wedge Y) \to Y$

(3) $X \to (X \vee Y)$

(4) $Y \to (X \vee Y)$

(5) $(\neg\neg X) \to X$

(6) $X \to (Y \to X)$

(7) $X \to (Y \to (X \wedge Y))$

(8) $((X \to Y) \wedge (X \to \neg Y)) \to \neg X$

(9) $((X \to Z) \wedge (Y \to Z)) \to ((X \vee Y) \to Z)$

(10) $((X \to Y) \wedge (X \to (Y \to Z))) \to (X \to Z)$

Rule of inference:

- (Modus Ponens) $\dfrac{X,\ X \to Y}{Y}$

Deriving other tautologies from the axioms.

Example 1. Derive $A \to A$ from axiom system 1.

1. $(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$
   axiom (2), replace X with A, Y with $B \to A$, and Z with A
2. $A \to ((B \to A) \to A)$ — axiom (1), replace X with A and Y with $B \to A$
3. $(A \to (B \to A)) \to (A \to A)$ — Modus Ponens, 1, 2
4. $A \to (B \to A)$ — axiom 1, replace X with A and Y with B
5. $A \to A$ — Modus Ponens, 3, 4

Example 2. Derive $(A \wedge B) \to (B \wedge A)$ from axiom system 2.

1. $(((A \wedge B) \to A) \wedge ((A \wedge B) \to (A \to (B \wedge A))) \to ((A \wedge B) \to (B \wedge A))$
axiom (10), replace X with $(A \wedge B)$, Y with A, and Z with $(B \wedge A)$
2. $(A \wedge B) \to A$ axiom (1)
3. $(((A \wedge B) \to B) \wedge ((A \wedge B) \to (B \to (A \to (B \wedge A))))) \to ((A \wedge B) \to (A \to (B \wedge A)))$
axiom (10), replace X with $(A \wedge B)$, Y with B, and Z with $[A \to (B \wedge A)]$
4. $(A \wedge B) \to B$ axiom (2)
5. $(B \to (A \to (B \wedge A))) \to ((A \wedge B) \to (B \to (A \to (B \wedge A))))$
axiom (6), replace X with $(B \to (A \to (B \wedge A)))$ and Y with $(A \wedge B)$
6. $B \to (A \to (B \wedge A))$ axiom (7)
7. $(A \wedge B) \to (B \to (A \to (B \wedge A)))$ Modus Ponens, 6, 5
8. $((A \wedge B) \to B) \to (((A \wedge B) \to (B \to (A \to (B \wedge A)))) \to (((A \wedge B) \to B) \wedge ((A \wedge B) \to (B \to (A \to (B \wedge A))))))$
axiom (7), replace X with $((A \wedge B) \to B)$ and Y with $((A \wedge B) \to (B \to (A \to (B \wedge A))))$
9. $((A \wedge B) \to (B \to (A \to (B \wedge A)))) \to (((A \wedge B) \to B) \wedge ((A \wedge B) \to (B \to (A \to (B \wedge A)))))$
Modus Ponens, 4, 8
10. $((A \wedge B) \to B) \wedge ((A \wedge B) \to (B \to (A \to (B \wedge A))))$
Modus Ponens, 7, 9
11. $(A \wedge B) \to (A \to (B \wedge A))$ Modus Ponens, 3, 10
12. $((A \wedge B) \to A) \to (((A \wedge B) \to (A \to (B \wedge A))) \to (((A \wedge B) \to A) \wedge ((A \wedge B) \to (A \to (B \wedge A)))))$
axiom (7), replace X with $((A \wedge B) \to A)$ and Y with $((A \wedge B) \to (A \to (B \wedge A)))$
13. $((A \wedge B) \to (A \to (B \wedge A))) \to (((A \wedge B) \to A) \wedge ((A \wedge B) \to (A \to (B \wedge A))))$ Modus Ponens, 2, 12
14. $((A \wedge B) \to A) \wedge ((A \wedge B) \to (A \to (B \wedge A)))$ Modus Ponens, 11, 13
15. $(A \wedge B) \to (B \wedge A)$ Modus Ponens, 1, 14