EXERCISES FOR CHAPTER 2

Section 2.1: Statements

2.1. Which of the following sentences are statements? For those that are, indicate the truth value.

(a) The integer 123 is prime.
(b) The integer 0 is even.
(c) Is $5 \times 2 = 10$?
(d) $x^2 - 4 = 0$.
(e) Multiply $5x + 2$ by 3.
(f) $5x + 3$ is an odd integer.
(g) What an impossible question!

2.2. Consider the sets A, B, C, and D below. Which of the following statements are true? Give an explanation for each false statement.

$$A = \{1, 4, 7, 10, 13, 16, \ldots\} \quad C = \{x \in \mathbf{Z} : x \text{ is prime and } x \neq 2\}$$
$$B = \{x \in \mathbf{Z} : x \text{ is odd}\} \quad D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$$

(a) $25 \in A$, (b) $33 \in D$, (c) $22 \notin A \cup D$, (d) $C \subseteq B$, (e) $\emptyset \in B \cap D$, (f) $53 \notin C$.

2.3. Which of the following statements are true? Give an explanation for each false statement.
(a) $\emptyset \in \emptyset$ (b) $\emptyset \in \{\emptyset\}$ (c) $\{1, 3\} = \{3, 1\}$
(d) $\emptyset = \{\emptyset\}$ (e) $\emptyset \subset \{\emptyset\}$ (f) $1 \subseteq \{1\}$.

2.4. The following is an open sentence over the domain $\mathbf{R}$:

$$P(x) : x(x - 1) = 6.$$

(a) For what values of x is $P(x)$ a true statement?
(b) For what values of x is $P(x)$ a false statement?

2.5. For the open sentence $P(x) : 3x - 2 > 4$ over the domain $\mathbf{Z}$, determine:

(a) the values of x for which $P(x)$ is true;
(b) the values of x for which $P(x)$ is false.

2.6. For the open sentence $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$, determine:

(a) all $A \in S$ for which $P(A)$ is true;
(b) all $A \in S$ for which $P(A)$ is false;
(c) all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$.

2.7. Let

$$P(n) : n \text{ and } n + 2 \text{ are primes.}$$

be an open sentence over the domain $\mathbf{N}$. Find six positive integers n for which $P(n)$ is true. If $n \in \mathbf{N}$ such that $P(n)$ is true, then the two integers $n, n + 2$ are called **twin primes**. It has been conjectured that there are infinitely many twin primes.

Section 2.2: The Negation of a Statement

2.8. State the negation of each of the following statements.

(a) $\sqrt{2}$ is a rational number.
(b) 0 is not a negative integer.
(c) 111 is a prime number.

2.9. Complete the truth table in Figure 2.16.

P	Q	$\sim P$	$\sim Q$
T	T		
T	F		
F	T		
F	F		

Figure 2.16 The truth table for Exercise 2.9.

P	Q	$\sim Q$	$P \wedge (\sim Q)$
T	T		
T	F		
F	T		
F	F		

Figure 2.17 The truth table for Exercise 2.12.

Section 2.3: The Disjunction and Conjunction of Statements

2.10. Let P: 15 is odd and Q : 21 is prime. State each of the following in words, and determine whether they are true or false. (a) $P \vee Q$ (b) $P \wedge Q$ (c) $(\sim P) \vee Q$ (d) $P \wedge (\sim Q)$.

2.11. For the sets $A = \{1, 2, \cdots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

$$P : A \subseteq B. \quad Q : |A - B| = 6.$$

Determine which of the following statements are true:
(a) $P \vee Q$ (b) $P \vee (\sim Q)$ (c) $P \wedge Q$
(d) $(\sim P) \wedge Q$ (e) $(\sim P) \vee (\sim Q)$.

2.12. Complete the truth table in Figure 2.17.

2.13. Let $S = \{1, 2, \ldots, 6\}$ and let

$$P(A) : A \cap \{2, 4, 6\} = \emptyset \text{ and } Q(A) : A \neq \emptyset.$$

be open sentences over the domain $\mathcal{P}(S)$.

(a) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \wedge Q(A)$ is true.
(b) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \vee (\sim Q(A))$ is true.
(c) Determine all $A \in \mathcal{P}(S)$ for which $(\sim P(A)) \wedge (\sim Q(A))$ is true.

Section 2.4: The Implication

2.14. Consider the statements P : 17 is even and Q : 19 is prime. Write each of the following statements in words, and indicate whether it is true or false.
(a) $\sim P$ (b) $P \vee Q$ (c) $P \wedge Q$ (d) $P \Rightarrow Q$.

2.15. For statements P and Q, construct a truth table for $(P \Rightarrow Q) \Rightarrow (\sim P)$.

2.16. Consider the statements P : $\sqrt{2}$ is rational and Q : 22/7 is rational. Write each of the following statements in words and indicate whether it is true or false.
(a) $P \Rightarrow Q$ (b) $Q \Rightarrow P$ (c) $(\sim P) \Rightarrow (\sim Q)$ (d) $(\sim Q) \Rightarrow (\sim P)$.

2.17. Consider the statements:

$$P : \sqrt{2} \text{ is rational,} \quad Q : \tfrac{2}{3} \text{ is rational,} \quad R : \sqrt{3} \text{ is rational.}$$

Write each of the following statements in words and indicate whether the statement is true or false.

(a) $(P \wedge Q) \Rightarrow R$
(b) $(P \wedge Q) \Rightarrow (\sim R)$
(c) $((\sim P) \wedge Q) \Rightarrow R$
(d) $(P \vee Q) \Rightarrow (\sim R)$.

Section 2.5: More on Implications

2.18. Consider the open sentences $P(n) : 5n + 3$ is prime and $Q(n) : 7n + 1$ is prime over the domain **N**.

(a) State $P(n) \Rightarrow Q(n)$ in words.

(b) State $P(2) \Rightarrow Q(2)$ in words. Is this statement true or false?

(c) State $P(6) \Rightarrow Q(6)$ in words. Is this statement true or false?

2.19. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine the truth value of $P(x) \Rightarrow Q(x)$ for each $x \in S$.

(a) $P(x) : |x| = 4;\ \ Q(x) : x = 4;\ \ S = \{-4, -3, 1, 4, 5\}$.

(b) $P(x) : x^2 = 16;\ \ Q(x) : |x| = 4;\ \ S = \{-6, -4, 0, 3, 4, 8\}$.

(c) $P(x) : x > 3;\ \ Q(x) : 4x - 1 > 12;\ \ S = \{0, 2, 3, 4, 6\}$.

2.20. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.

(a) $P(x) : x - 3 = 4;\ \ Q(x) : x \geq 8;\ \ S = \mathbf{R}$.

(b) $P(x) : x^2 \geq 1;\ \ Q(x) : x \geq 1;\ \ S = \mathbf{R}$.

(c) $P(x) : x^2 \geq 1;\ \ Q(x) : x \geq 1;\ \ S = \mathbf{N}$.

(d) $P(x) : x \in [-1, 2];\ \ Q(x) : x^2 \leq 2;\ \ S = [-1, 1]$.

2.21. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is **Z**. Determine the truth value of $P(x, y) \Rightarrow Q(x, y)$ for the given values of x and y.

(a) $P(x, y) : x^2 - y^2 = 0$ and $Q(x, y) : x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.

(b) $P(x, y) : |x| = |y|$ and $Q(x, y) : x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.

(c) $P(x, y) : x^2 + y^2 = 1$ and $Q(x, y) : x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

Section 2.6: The Biconditional

2.22. Let P : 18 is odd and Q : 25 is even. State $P \Leftrightarrow Q$ in words. Is $P \Leftrightarrow Q$ true or false?

2.23. Consider the open sentences:

$$P(x) : x = -2. \text{ and } Q(x) : x^2 = 4.$$

over the domain $S = \{-2, 0, 2\}$. State each of the following in words and determine all values of $x \in S$ for which the resulting statements are true.

(a) $\sim P(x)$ (b) $P(x) \vee Q(x)$ (c) $P(x) \wedge Q(x)$ (d) $P(x) \Rightarrow Q(x)$
(e) $Q(x) \Rightarrow P(x)$ (f) $P(x) \Leftrightarrow Q(x)$.

2.24. For the following open sentences $P(x)$ and $Q(x)$ over a domain S, determine all values of $x \in S$ for which the biconditional $P(x) \Leftrightarrow Q(x)$ is true.

(a) $P(x) : |x| = 4;\ \ Q(x) : x = 4;\ \ S = \{-4, -3, 1, 4, 5\}$.

(b) $P(x) : x \geq 3;\ \ Q(x) : 4x - 1 > 12;\ \ S = \{0, 2, 3, 4, 6\}$.

(c) $P(x) : x^2 = 16;\ \ Q(x) : x^2 - 4x = 0;\ \ S = \{-6, -4, 0, 3, 4, 8\}$.

2.25. Let $P(x) : x$ is odd. and $Q(x) : x^2$ is odd. be open sentences over the domain **Z**. State $P(x) \Leftrightarrow Q(x)$ in two ways: (1) using "if and only if" and (2) using "necessary and sufficient".

2.26. For the open sentences $P(x) : |x - 3| < 1$ and $Q(x) : x \in (2, 4)$. over the domain **R**, state the biconditional $P(x) \Leftrightarrow Q(x)$ in two different ways.

2.27. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbf{Z}$. Determine the truth value of $P(x, y) \Leftrightarrow Q(x, y)$ for the given values of x and y.

(a) $P(x, y) : x^2 - y^2 = 0$ and $Q(x, y) : x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.

(b) $P(x, y) : |x| = |y|$ and $Q(x, y) : x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.

(c) $P(x, y) : x^2 + y^2 = 1$ and $Q(x, y) : x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

2.28. Let $S = \{1, 2, 3\}$. Consider the following open sentences over the domain S:

$$P(n) : \frac{(n+4)(n+5)}{2} \text{ is odd.}$$
$$Q(n) : 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}.$$

Determine three distinct elements a, b, c in S such that $P(a) \Rightarrow Q(a)$ is false, $Q(b) \Rightarrow P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.

2.29. Let $S = \{1, 2, 3, 4\}$. Consider the following open sentences over the domain S:

$$P(n) : \frac{n(n-1)}{2} \text{ is even.}$$
$$Q(n) : 2^{n-2} - (-2)^{n-2} \text{ is even.}$$
$$R(n) : 5^{n-1} + 2^n \text{ is prime.}$$

Determine four distinct elements a, b, c, d in S such that

(i) $P(a) \Rightarrow Q(a)$ is false; (ii) $Q(b) \Rightarrow P(b)$ is true;
(iii) $P(c) \Leftrightarrow R(c)$ is true; (iv) $Q(d) \Leftrightarrow R(d)$ is false.

Section 2.7: Tautologies and Contradictions

2.30. For statements P and Q, show that $P \Rightarrow (P \vee Q)$ is a tautology.

2.31. For statements P and Q, show that $(P \wedge \sim Q) \wedge (P \wedge Q)$ is a contradiction.

2.32. For statements P and Q, show that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology. Then state $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ in words. (This is an important logical argument form, called **modus ponens**.)

2.33. For statements P, Q, and R, show that $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology. Then state this compound statement in words. (This is another important logical argument form, called **syllogism**.)

Section 2.8: Logical Equivalence

2.34. For statements P and Q, the implication $(\sim P) \Rightarrow (\sim Q)$ is called the **inverse** of the implication $P \Rightarrow Q$.

(a) Use a truth table to show that these statements are not logically equivalent.

(b) Find another implication that is logically equivalent to $\sim P \Rightarrow \sim Q$ and verify your answer.

2.35. Let P and Q be statements.

(a) Is $\sim(P \vee Q)$ logically equivalent to $(\sim P) \vee (\sim Q)$? Explain.

(b) What can you say about the biconditional $\sim(P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$?

2.36. For statements P, Q, and R, use a truth table to show that each of the following pairs of statements are logically equivalent.

(a) $(P \wedge Q) \Leftrightarrow P$ and $P \Rightarrow Q$.

(b) $P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$.

2.37. For statements P and Q, show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent.

2.38. For statements P, Q, and R, show that $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent.

Section 2.9: Some Fundamental Properties of Logical Equivalence

2.39. Verify the following laws stated in Theorem 2.18:

(a) Let P, Q, and R be statements. Then

$$P \vee (Q \wedge R) \text{ and } (P \vee Q) \wedge (P \vee R) \text{ are logically equivalent.}$$

(b) Let P and Q be statements. Then

$$\sim(P \vee Q) \text{ and } (\sim P) \wedge (\sim Q) \text{ are logically equivalent.}$$

2.40. Write negations of the following open sentences:

(a) Either $x = 0$ or $y = 0$.
(b) The integers a and b are both even.

2.41. Consider the implication: If x and y are even, then xy is even.

(a) State the implication using "only if ".
(b) State the converse of the implication.
(c) State the implication as a disjunction (see Theorem 2.17).
(d) State the negation of the implication as a conjunction (see Theorem 2.21(a)).

2.42. For a real number x, let $P(x) : x^2 = 2$ and $Q(x) : x = \sqrt{2}$. State the negation of the biconditional $P \Leftrightarrow Q$ in words (see Theorem 2.21(b)).

Section 2.10: Quantified Statements

2.43. Let S denote the set of odd integers, and let

$$P(x) : x^2 + 1 \text{ is even.} \quad \text{and} \quad Q(x) : x^2 \text{ is even.}$$

be open sentences over the domain S. State $\forall x \in S, P(x)$ and $\exists x \in S, Q(x)$ in words.

2.44. Define an open sentence $R(x)$ over some domain S and then state $\forall x \in S, R(x)$ and $\exists x \in S, R(x)$ in words.

2.45. State the negations of the following quantified statements, where all sets are subsets of some universal set U:

(a) For every set A, $A \cap \overline{A} = \emptyset$.
(b) There exists a set A such that $\overline{A} \subseteq A$.

2.46. State the negations of the following quantified statements:

(a) For every rational number r, the number $1/r$ is rational.
(b) There exists a rational number r such that $r^2 = 2$.

2.47. Let $P(n) : (5n - 6)/3$ is an integer. be an open sentence over the domain $\mathbf{Z}$. Determine, with explanations, whether the following statements are true:

(a) $\forall n \in \mathbf{Z}, P(n)$.
(b) $\exists n \in \mathbf{Z}, P(n)$.

2.48. Determine the truth value of each of the following statements.

(a) $\exists x \in \mathbf{R}, x^2 - x = 0$.
(b) $\forall n \in \mathbf{N}, n + 1 \geq 2$.
(c) $\forall x \in \mathbf{R}, \sqrt{x^2} = x$.
(d) $\exists x \in \mathbf{Q}, 3x^2 - 27 = 0$.

(e) $\exists x \in \mathbf{R}, \exists y \in \mathbf{R}, x + y + 3 = 8$.
(f) $\forall x, y \in \mathbf{R}, x + y + 3 = 8$.
(g) $\exists x, y \in \mathbf{R}, x^2 + y^2 = 9$.
(h) $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}, x^2 + y^2 = 9$.

2.49. The statement

For every integer m, either $m \leq 1$ or $m^2 \geq 4$.

can be expressed using a quantifier as:

$$\forall m \in \mathbf{Z}, m \leq 1 \text{ or } m^2 \geq 4.$$

Do this for the statements in parts (a) and (b).

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.
(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.
(c) Express in words the negations of the statements in (a) and (b).
(d) Using quantifiers, express in symbols the negations of the statements in both (a) and (b).

2.50. Consider the open sentence

$$P(x, y, z):\ (x-1)^2 + (y-2)^2 + (z-2)^2 > 0.$$

where the domain of each of the variables x, y and z is $\mathbf{R}$.

(a) Express the quantified statement $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}, \forall z \in \mathbf{R}, P(x, y, z)$ in words.
(b) Is the quantified statement in (a) true or false? Explain.
(c) Express the negation of the quantified statement in (a) in symbols.
(d) Express the negation of the quantified statement in (a) in words.
(e) Is the negation of the quantified statement in (a) true or false? Explain.

2.51. Consider the quantified statement

For every $s \in S$ and $t \in S$, $st - 2$ is prime.

where the domain of the variables s and t is $S = \{3, 5, 11\}$.

(a) Express this quantified statement in symbols.
(b) Is the quantified statement in (a) true or false? Explain.
(c) Express the negation of the quantified statement in (a) in symbols.
(d) Express the negation of the quantified statement in (a) in words.
(e) Is the negation of the quantified statement in (a) true or false? Explain.

Section 2.11: Characterizations of Statements

2.52. Give a definition of each of the following, and then state a characterization of each.

(a) two lines in the plane are perpendicular
(b) a rational number

2.53. Define an integer n to be odd if n is not even. State a characterization of odd integers.

2.54. Define a triangle to be isosceles if it has two equal sides. Which of the following statements are characterizations of isosceles triangles? If a statement is not a characterization of isosceles triangles, then explain why.

(a) If a triangle is equilateral, then it is isosceles.
(b) A triangle T is isosceles if and only if T has two equal sides.

(c) If a triangle has two equal sides, then it is isosceles.
(d) A triangle T is isosceles if and only if T is equilateral.
(e) If a triangle has two equal angles, then it is isosceles.
(f) A triangle T is isosceles if and only if T has two equal angles.

2.55. By definition, a right triangle is a triangle one of whose angles is a right angle. Also, two angles in a triangle are complementary if the sum of their degrees is 90°. Which of the following statements are characterizations of a right triangle? If a statement is not a characterization of a right triangle, then explain why.

(a) A triangle is a right triangle if and only if two of its sides are perpendicular.
(b) A triangle is a right triangle if and only if it has two complementary angles.
(c) A triangle is a right triangle if and only if its area is half of the product of the lengths of some pair of its sides.
(d) A triangle is a right triangle if and only if the square of the length of its longest side equals the sum of the squares of the lengths of the two smallest sides.
(e) A triangle is a right triangle if and only if twice the area of the triangle equals the area of some rectangle.

ADDITIONAL EXERCISES FOR CHAPTER 2

2.56. Construct a truth table for $P \wedge (Q \Rightarrow \sim P)$.

2.57. Given that the implication $(Q \vee R) \Rightarrow \sim P$ is false and Q is false, determine the truth values of R and P.

2.58. Find a compound statement involving the component statements P and Q that has the truth table given in Figure 2.18.

2.59. Determine the truth value of each of the following quantified statements:

(a) $\exists x \in \mathbf{R}, x^2 - x = 0$.
(b) $\forall n \in \mathbf{N}, n + 1 \geq 2$.
(c) $\forall x \in \mathbf{R}, \sqrt{x^2} = x$.
(d) $\exists x \in \mathbf{Q}, \frac{1}{x^2} = \frac{1}{2}$.
(e) $\exists x, y \in \mathbf{R}, x + y + 3 = 8$.
(f) $\forall x, y \in \mathbf{R}, x + y + 3 = 8$.

2.60. Rewrite each of the implications below using (1) only if and (2) sufficient.

(a) If a function f is differentiable, then f is continuous.
(b) If $x = -5$, then $x^2 = 25$.

P	Q	$\sim Q$	
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Figure 2.18 Truth table for Exercise 2.58.

2.61. Let

$$P(n) : n^2 - n + 5 \text{ is a prime.}$$

be an open sentence over a domain S.

(a) Determine the truth values of the quantified statements $\forall n \in S,\ P(n)$ and $\exists n \in S,\ {\sim}P(n)$ for $S = \{1, 2, 3, 4\}$.

(b) Determine the truth values of the quantified statements $\forall n \in S,\ P(n)$ and $\exists n \in S,\ {\sim}P(n)$ for $S = \{1, 2, 3, 4, 5\}$.

(c) How are the statements in (a) and (b) related?

2.62. (a) For statements P, Q, and R, show that

$$((P \wedge Q) \Rightarrow R) \equiv ((P \wedge ({\sim}R)) \Rightarrow ({\sim}Q)).$$

(b) For statements P, Q, and R, show that

$$((P \wedge Q) \Rightarrow R) \equiv (Q \wedge ({\sim}R) \Rightarrow ({\sim}P)).$$

2.63. For a fixed integer n, use Exercise 2.62 to restate the following implication in two different ways:

If n is a prime and $n > 2$, then n is odd.

2.64. For fixed integers m and n, use Exercise 2.62 to restate the following implication in two different ways:

If m is even and n is odd, then $m + n$ is odd.

2.65. For a real valued function f and a real number x, use Exercise 2.62 to restate the following implication in two different ways:

If $f'(x) = 3x^2 - 2x$ and $f(0) = 4$, then $f(x) = x^3 - x^2 + 4$.

2.66. For the set $S = \{1, 2, 3\}$, give an example of three open sentences $P(n)$, $Q(n)$, and $R(n)$, each over the domain S, such that (1) each of $P(n)$, $Q(n)$, and $R(n)$ is a true statement for exactly two elements of S, (2) all of the implications $P(1) \Rightarrow Q(1)$, $Q(2) \Rightarrow R(2)$, and $R(3) \Rightarrow P(3)$ are true, and (3) the converse of each implication in (2) is false.

2.67. Do there exist a set S of cardinality 2 and a set $\{P(n), Q(n), R(n)\}$ of three open sentences over the domain S such that the implications $P(a) \Rightarrow Q(a)$, $Q(b) \Rightarrow R(b)$, and $R(c) \Rightarrow P(c)$ are true, where $a, b, c \in S$, and (2) the converses of the implications in (1) are false? Necessarily, at least two of these elements a, b, and c of S are equal.

2.68. Let $A = \{1, 2, \ldots, 6\}$ and $B = \{1, 2, \ldots, 7\}$. For $x \in A$, let $P(x) : 7x + 4$ is odd. For $y \in B$, let $Q(y) : 5y + 9$ is odd. Let

$$S = \{(P(x), Q(y)) : x \in A, y \in B, P(x) \Rightarrow Q(y) \text{ is false}\}.$$

What is $|S|$?