EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Which of the following are sets?

(a) $1, 2, 3$
(b) $\{1, 2\}, 3$
(c) $\{\{1\}, 2\}, 3$
(d) $\{1, \{2\}, 3\}$
(e) $\{1, 2, a, b\}$

1.2. Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{1, 2, 3\}$
(b) $B = \{0, 1, 2, 3\}$
(c) $C = \{-2, -1\}$.
(d) $D = \{-2, 2, 3\}$.

1.3. Determine the cardinality of each of the following sets:

(a) $A = \{1, 2, 3, 4, 5\}$
(b) $B = \{0, 2, 4, \ldots, 20\}$
(c) $C = \{25, 26, 27, \ldots, 75\}$

(d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$
(e) $E = \{\emptyset\}$
(f) $F = \{2, \{2, 3, 4\}\}$

1.4. Write each of the following sets by listing its elements within braces.

(a) $A = \{n \in \mathbf{Z} : -4 < n \leq 4\}$
(b) $B = \{n \in \mathbf{Z} : n^2 < 5\}$
(c) $C = \{n \in \mathbf{N} : n^3 < 100\}$
(d) $D = \{x \in \mathbf{R} : x^2 - x = 0\}$
(e) $E = \{x \in \mathbf{R} : x^2 + 1 = 0\}$

1.5. Write each of the following sets in the form $\{x \in \mathbf{Z} : p(x)\}$, where $p(x)$ is a property concerning x.

(a) $A = \{-1, -2, -3, \ldots\}$
(b) $B = \{-3, -2, \ldots, 3\}$
(c) $C = \{-2, -1, 1, 2\}$

1.6. The set $E = \{2x : x \in \mathbf{Z}\}$ can be described by listing its elements, namely $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. List the elements of the following sets in a similar manner.

(a) $A = \{2x + 1 : x \in \mathbf{Z}\}$
(b) $B = \{4n : n \in \mathbf{Z}\}$
(c) $C = \{3q + 1 : q \in \mathbf{Z}\}$

1.7. The set $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbf{Z}\} = \{2x : x \in \mathbf{Z}\}$. Describe the following sets in a similar manner.

(a) $A = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$
(b) $B = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$
(c) $C = \{1, 8, 27, 64, 125, \ldots\}$

Section 1.2: Subsets

1.8. Give examples of three sets A, B, and C such that

(a) $A \subseteq B \subset C$.
(b) $A \in B$, $B \in C$, and $A \notin C$.
(c) $A \in B$ and $A \subset C$.

1.9. Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.

1.10. Which of the following sets are equal?

$A = \{n \in \mathbf{Z} : |n| < 2\}$ $\quad D = \{n \in \mathbf{Z} : n^2 \leq 1\}$
$B = \{n \in \mathbf{Z} : n^3 = n\}$ $\quad E = \{-1, 0, 1\}$
$C = \{n \in \mathbf{Z} : n^2 \leq n\}$

1.11. For a universal set $U = \{1, 2, \ldots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets.

1.12. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for

(a) $A = \{1, 2\}$.
(b) $A = \{\emptyset, 1, \{a\}\}$.

1.13. Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.

1.14. Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

1.15. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \emptyset, \{\emptyset\}\}$.

1.16. Give an example of a set S such that

(a) $S \subseteq \mathcal{P}(\mathbf{N})$
(b) $S \in \mathcal{P}(\mathbf{N})$
(c) $S \subseteq \mathcal{P}(\mathbf{N})$ and $|S| = 5$.
(d) $S \in \mathcal{P}(\mathbf{N})$ and $|S| = 5$.

Section 1.3: Set Operations

1.17. Let $U = \{1, 3, \ldots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$. Determine the following:

(a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, (d) $B - A$, (e) $\overline{A}$, (f) $A \cap \overline{B}$.

1.18. Give examples of three sets A, B, and C such that

(a) $A \in B$, $A \subseteq C$, and $B \not\subseteq C$.
(b) $B \in A$, $B \subset C$, and $A \cap C \neq \emptyset$.
(c) $A \in B$, $B \subseteq C$, and $A \not\subseteq C$.

1.19. Give examples of three sets A, B, and C such that $B \neq C$ but $B - A = C - A$.

1.20. Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$. Draw the accompanying Venn diagram.

1.21. Let U be a universal set and let A and B be two subsets of U. Draw a Venn diagram for each of the following sets.

(a) $\overline{A \cup B}$ (b) $\overline{A} \cap \overline{B}$ (c) $\overline{A \cap B}$ (d) $\overline{A} \cup \overline{B}$

What can you say about parts (a) and (b)? parts (c) and (d)?

1.22. Give an example of a universal set U, two sets A and B, and an accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = |\overline{A \cup B}| = 2$.

1.23. Let A, B, and C be nonempty subsets of a universal set U. Draw a Venn diagram for each of the following set operations.

(a) $(C - B) \cup A$
(b) $C \cap (A - B)$

1.24. Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.

(a) Determine which of the following are elements of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
(b) Determine $|A|$.
(c) Determine which of the following are subsets of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
For (d)–(i), determine the indicated sets.
(d) $\emptyset \cap A$
(e) $\{\emptyset\} \cap A$
(f) $\{\emptyset, \{\emptyset\}\} \cap A$
(g) $\emptyset \cup A$
(h) $\{\emptyset\} \cup A$
(i) $\{\emptyset, \{\emptyset\}\} \cup A$.

Section 1.4: Indexed Collections of Sets

1.25. Give examples of a universal set U and sets A, B, and C such that each of the following sets contains exactly one element: $A \cap B \cap C$, $(A \cap B) - C$, $(A \cap C) - B$, $(B \cap C) - A$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$, $\overline{A \cup B \cup C}$. Draw the accompanying Venn diagram.

1.26. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r-1, r+1]$, and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine

(a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$

(b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$

(c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

1.27. Let $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, $C = \{2, 3, 4\}$, and $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ and $\bigcap_{X \in S} X$.

1.28. For a real number r, define S_r to be the interval $[r-1, r+2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

1.29. Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let A_α consist of α and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

1.30. For each of the following collections of sets, define a set A_n for each $n \in \mathbf{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbf{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.

(a) $\{[1, 2+1), [1, 2+1/2), [1, 2+1/3), \ldots\}$

(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$

1.31. For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbf{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.

(a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1]$.

(b) $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup_{n=1}^{\infty} A_n = \mathbf{Z}$.

Section 1.5: Partitions of Sets

1.32. Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$

(b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$

(c) $S_3 = \{A\}$

(d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$

(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$

1.33. Which of the following sets are partitions of $A = \{1, 2, 3, 4, 5\}$?

(a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$

(b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}, \{2, 1\}\}$

(c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$

(d) $S_4 = A$

1.34. Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition S of A such that $|S| = 3$.

1.35. Give an example of a set A with $|A| = 4$ and two disjoint partitions S_1 and S_2 of A with $|S_1| = |S_2| = 3$.

1.36. Give an example of three sets A, S_1, and S_2 such that S_1 is a partition of A, S_2 is a partition of S_1, and $|S_2| < |S_1| < |A|$.

1.37. Give an example of a partition of $\mathbf{Q}$ into three subsets.

1.38. Give an example of a partition of $\mathbf{N}$ into three subsets.

1.39. Give an example of a partition of $\mathbf{Z}$ into four subsets.

1.40. Let $A = \{1, 2, \ldots, 12\}$. Give an example of a partition S of A satisfying the following requirements: (i) $|S| = 5$, (ii) T is a subset of S such that $|T| = 4$ and $|\cup_{X \in T} X| = 10$, and (iii) there is no element $B \in S$ such that $|B| = 3$.

Section 1.6: Cartesian Products of Sets

1.41. Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Determine $A \times B$.

1.42. Let $A = \{1, \{1\}, \{\{1\}\}\}$. Determine $A \times A$.

1.43. For $A = \{a, b\}$. Determine $A \times \mathcal{P}(A)$.

1.44. For $A = \{\emptyset, \{\emptyset\}\}$. Determine $A \times \mathcal{P}(A)$.

1.45. For $A = \{1, 2\}$ and $B = \{\emptyset\}$, determine $A \times B$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.

1.46. Describe the graph of the circle whose equation is $x^2 + y^2 = 4$ as a subset of $\mathbf{R} \times \mathbf{R}$.

1.47. List the elements of the set $S = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : |x| + |y| = 3\}$. Plot the corresponding points in the Euclidean x-y plane.

ADDITIONAL EXERCISES FOR CHAPTER 1

1.48. Let $S = \{-10, -9, \ldots, 9, 10\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{-10, -9, \ldots, -1, 1, \ldots, 9, 10\}$
(b) $B = \{-10, -9, \ldots, -1, 0\}$
(c) $C = \{-5, -4, \ldots, 7\}$
(d) $D = \{-10, -9, \ldots, 4, 6, 7, \ldots, 10\}$

1.49. Describe each of the following sets by listing its elements within braces.

(a) $\{x \in \mathbf{Z} : x^3 - 4x = 0\}$
(b) $\{x \in \mathbf{R} : |x| = -1\}$
(c) $\{m \in \mathbf{N} : 2 < m \leq 5\}$
(d) $\{n \in \mathbf{N} : 0 \leq n \leq 3\}$
(e) $\{k \in \mathbf{Q} : k^2 - 4 = 0\}$
(f) $\{k \in \mathbf{Z} : 9k^2 - 3 = 0\}$
(g) $\{k \in \mathbf{Z} : 1 \leq k^2 \leq 10\}$

1.50. Determine the cardinality of each of the following sets.

(a) $A = \{1, 2, 3, \{1, 2, 3\}, 4, \{4\}\}$
(b) $B = \{x \in \mathbf{R} : |x| = -1\}$
(c) $C = \{m \in \mathbf{N} : 2 < m \leq 5\}$
(d) $D = \{n \in \mathbf{N} : n < 0\}$
(e) $E = \{k \in \mathbf{N} : 1 \leq k^2 \leq 100\}$
(f) $F = \{k \in \mathbf{Z} : 1 \leq k^2 \leq 100\}$

1.51. For $A = \{-1, 0, 1\}$ and $B = \{x, y\}$, determine $A \times B$.

1.52. Let $U = \{1, 2, 3\}$ be the universal set, and let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{1, 3\}$. Determine the following.

(a) $(A \cup B) - (B \cap C)$
(b) $\overline{A}$
(c) $\overline{B \cup C}$
(d) $A \times B$

1.53. Let $A = \{1, 2, \ldots, 10\}$. Give an example of two sets S and B such that $S \subseteq \mathcal{P}(A)$, $|S| = 4$, $B \in S$, and $|B| = 2$.

1.54. For $A = \{1\}$ and $C = \{1, 2\}$, give an example of a set B such that $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$.

1.55. Give examples of two sets A and B such that

(a) $A \cap \mathcal{P}(A) \in B$
(b) $\mathcal{P}(A) \subseteq A \cup B$.

1.56. Which of the following sets are equal?

$A = \{n \in \mathbf{Z} : -4 \le n \le 4\}$ $\quad D = \{x \in \mathbf{Z} : x^3 = 4x\}$
$B = \{x \in \mathbf{N} : 2x + 2 = 0\}$ $\quad E = \{-2, 0, 2\}$
$C = \{x \in \mathbf{Z} : 3x - 2 = 0\}$

1.57. Let A and B be sets in some unknown universal set U. Suppose that $\overline{A} = \{3, 8, 9\}$, $A - B = \{1, 2\}$, $B - A = \{8\}$, and $A \cap B = \{5, 7\}$. Determine U, A, and B.

1.58. Let I denote the interval $[0, \infty)$. For each $r \in I$, define

$$A_r = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = r^2\right\},$$
$$B_r = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 \le r^2\right\},$$
$$C_r = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 > r^2\right\}.$$

(a) Determine $\bigcup_{r \in I} A_r$ and $\bigcap_{r \in I} A_r$.
(b) Determine $\bigcup_{r \in I} B_r$ and $\bigcap_{r \in I} B_r$.
(c) Determine $\bigcup_{r \in I} C_r$ and $\bigcap_{r \in I} C_r$.

1.59. Give an example of four sets A_1, A_2, A_3, A_4 such that $|A_i \cap A_j| = |i - j|$ for every two integers i and j with $1 \le i < j \le 4$.

1.60. (a) Give an example of two problems suggested by Exercise 1.59 (above).
(b) Solve one of the problems in (a).

1.61. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, and $C = \{1, 2, 3, 4, 5\}$. For the sets S and T described below, explain whether $|S| < |T|$, $|S| > |T|$, or $|S| = |T|$.

(a) Let B be the universal set and let S be the set of all subsets X of B for which $|X| \ne |\overline{X}|$. Let T be the set of 2-element subsets of C.
(b) Let S be the set of all partitions of the set A and let T be the set of 4-element subsets of C.
(c) Let $S = \{(b, a) : b \in B, a \in A, a + b \text{ is odd}\}$ and let T be the set of all nonempty proper subsets of A.

1.62. Give an example of a set $A = \{1, 2, \ldots, k\}$ for a smallest $k \in \mathbf{N}$ having subsets A_1, A_2, A_3 such that $|A_i - A_j| = |A_j - A_i| = |i - j|$ for every two integers i and j with $1 \le i < j \le 3$.