MATH 110 **Practice Questions for Test 3**

1. For each statement, state whether it is true or false.
   (a) Unlike in the classical logic, in modal logic we can't say that a formula has a truth value (true or false) given the truth values of its components (propositional variables).
   (b) The formula $\Box P \to P$ is a tautology.
   (c) The operation $\Diamond$ is defined by $\Diamond P = \neg\Box\neg P$.
   (d) Necessitation rule says that if we derived $\Box P$, then we derived P.
   (e) Axiom T is derivable from axioms K and 4.
   (f) Axiom D is derivable in $S4$.
2. For $X = \{a, b, c, d\}$, Determine which of the following are topologies on X. For those that are not, identify all axioms (out of the four axioms in the definition of a topological space) that do not hold.
   (a) $\tau = \{\emptyset, \{a\}, X\}$
   (b) $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$
   (c) $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, X\}$
   (d) $\tau = \{\emptyset, \{a, b\}, \{a, c\}, X\}$
3. Consider the set $\mathbb{R}$ with the usual topology. For each subset of $\mathbb{R}$ given below,
   - determine whether it is open, closed, both, or neither; and
   - find its interior and closure.

   (a) $\{1, 2, 3\}$
   (b) $[2, 3) \cap \mathbb{Q}$
   (c) $[2, 3] \cup [4, \infty)$
   (d) $(2, 3) \cup (3, 4)$
4. Consider $\mathbb{R}$ with the usual topology, and the following interpretation: $f(P) = (0, 3]$, $f(Q) = \{0, 1\} \cup [2, 4)$. Find the following:
   (a) $f(\Box Q)$
   (b) $f(\Diamond Q)$
   (c) $f(\neg\Box P)$
   (d) $f(P \wedge \Box Q)$
   (e) $f(\Box P \vee Q)$
   (f) $f(P \to \Diamond P)$

5. Give an example of a topological space in which the formula $\Diamond\Box P \vee \Diamond\Box\neg P$ is not valid.

6. Complete the following proof of $\Box\Diamond\Box\Diamond P \to \Box\Diamond P$ from S4.

   (1) $\Box P \to P$ (axiom _____)

   (2) $\Box\Diamond Q \to \Diamond Q$ (substitution _______________)

   (3) _______________________________________ (contrapositive of step (2))

   (4) _______________________________________ (apply $\Box$ to step (3))

   (5) $\Box\neg\neg\Box\neg Q \to \Box\neg\Box\Diamond Q$ (use ___________________ in step (4))

   (6) _____________________________ (simplify double negation in step (5))

   (7) $\Box\neg Q \to \Box\Box\neg Q$ (axiom _____ with substitution _______________)

   (8) $\Box\neg Q \to \Box\neg\Box\Diamond Q$ (steps (6), (7), and transitivity)

   (9) $\neg\Box\neg\Box\Diamond Q \to \neg\Box\neg Q$ (___________________ of step (8))

   (10) _____________________________ (rewrite each $\neg\Box\neg$ as $\Diamond$ in step (9))

   (11) $\Box\Diamond\Box\Diamond Q \to \Box\Diamond Q$ (apply _____ to step (10))