Propositional functions of several variables and nested quantifiers

Propositional functions (open sentences) can be functions of two or more variables. In this case we can use two or more quantifiers with them.

It is important to realize that the order of quantifiers makes a difference.

Example 1. Let the propositional function $F(x, y)$ denote the statement that x and y are friends (let the domain of this function be the set of all people in the world).

The proposition $\forall x \exists y F(x, y)$ means that everybody has at least one friend. The propostition $\exists y \forall x F(x, y)$ means that there is a person who is friends with everybody.

Example 2. Let $P(x, y)$ denote the proposition "$x < y$" where x and y are real numbers (where the domain for both variables is the set of all real numbers). Then

- $\exists x \exists y P(x, y)$ means "there exist real numbers x and y such that $x < y$". This is a true statement. For example, $x = 3$ and $y = 4$ satisfy $x < y$.
- $\forall x \exists y P(x, y)$ means "for any real number x there exists a real number y such that $x < y$". This is also a true statement since for any x we can choose $y = x + 1$, and then the condition $x < y$ is satisfied.
- $\exists x \forall y P(x, y)$ means "there exists a real number x such that for any real number y we have $x < y$". This is a false statement since for any x, the number $y = x-1$ does not satisfy the condition $x < y$. Thus there doesn't exist an x for which *all* y will satisfy the given condition.
- $\forall x \forall y P(x, y)$ means "for any real numbers x and y we have $x < y$". This is also a false statement. For example, the pair $x = 3$, $y = 2$ does not satisfy the condition $x < y$.

Propositions with negations can always be written so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives), for example:

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Exercises

1. Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$$

   into English, where $C(x)$ is "x has a computer", $F(x,y)$ is "x and y are friends", and the domain for both x and y is the set of all students at your university.

2. Let $F(x,y)$ be statement "x can fool y", where the domain for both variables is the set of all people in the world. Use quantifiers to express each of the following statements:
   (a) Everybody can fool Fred.
   (b) There is no one who can fool everybody.
   (c) Everyone can be fooled by somebody.
   (d) Tim can fool exactly two people.
   (e) There is exactly one person whom everybody can fool.
   (f) No one can fool himself or herself.

3. Let $Q(x,y)$ be the statement "$x+y = x-y$", and the domain for both variables is the set of integers. Find the truth values of the following statements. Explain.
   (a) $Q(2,0)$
   (b) $\forall y Q(1,y)$
   (c) $\forall x \exists y Q(x,y)$
   (d) $\forall y \exists x Q(x,y)$
   (e) $\exists y \forall x Q(x,y)$

4. Rewrite each of the following statements so that negations appear only within predicates.
   (a) $\neg \forall x \forall y P(x,y)$
   (b) $\neg \forall y \forall x (P(x,y) \vee Q(x,y))$
   (c) $\neg \forall x (\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z))$

5. Express the definition of the limit $\lim_{x \to a} f(x) = L$ using quantifiers.

6. Express the definition of a convergent sequence $a_1, a_2, \ldots$ using quantifiers.

Solutions

1. Every student at my university has a computer or has a friend who has a computer.

2. (a) $\forall x F(x, Fred)$

   (b) $\neg\exists x\forall y F(x,y)$ or $\forall x\exists y\neg F(x,y)$

   (c) $\forall y\exists x F(x,y)$ or $\forall x\exists y F(y,x)$

   (d) $\exists x\exists! y(x \neq y \wedge F(Tim,x) \wedge F(Tim,y))$ or
   $\exists x\exists y(x \neq y \wedge F(Tim,x) \wedge F(Tim,y) \wedge (\forall z(F(Tim,z) \Rightarrow (z = x \vee z = y))))$

   (e) $\exists! y\forall x F(x,y)$ or $\exists! x\forall y F(y,x)$

   (f) $\neg\exists x F(x,x)$ or $\forall x\neg F(x,x)$

3. (a) True because $2+0 = 2-0$ is true.

   (b) False because "for every y, $1+y = 1-y$" is false: for example, if $y = 1$, then $2 \neq 0$.

   (c) True because for any x we can take $y = 0$, then $x+0 = x-0$ is true.

   (d) False because for example if $y = 1$, there is no x such that $x+1 = x-1$.

   (e) True because if $y = 0$, then for any x we have $x+0 = x-0$.

   Note. Statements (c) and (e) are not equivalent a priori! (c) says that for any x we can find a y such that $Q(x,y)$ is true. It is possible that we will find different values of y for different values of x. While (e) says that there is a value of y that works for any x.

4. (a) $\neg\forall x\forall y P(x,y) \equiv \exists x\neg\forall y P(x,y) \equiv \exists x\exists y\neg P(x,y)$

   (b) $\neg\forall y\forall x(P(x,y) \vee Q(x,y)) \equiv \exists y\neg\forall x(P(x,y) \vee Q(x,y)) \equiv$
   $\exists y\exists x\neg(P(x,y) \vee Q(x,y)) \equiv \exists y\exists x((\neg P(x,y)) \wedge (\neg Q(x,y)))$

   (c) $\neg\forall x(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z)) \equiv$
   $\exists x\neg(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z)) \equiv$
   $\exists x((\neg\exists y\forall z P(x,y,z)) \vee (\neg\exists z\forall y P(x,y,z))) \equiv$
   $\exists x((\forall y\neg\forall z P(x,y,z)) \vee (\forall z\neg\forall y P(x,y,z))) \equiv$
   $\exists x((\forall y\exists z\neg P(x,y,z)) \vee (\forall z\exists y\neg P(x,y,z)))$

5 The definition is as follows: the limit $\lim\limits_{x\to a} f(x) = L$ if for any positive ε there exists a positive δ such that for any $x \in \mathbb{R}$, $0 < |x-a| < \delta$ implies $|f(x)-L| < \varepsilon$. Now we rewrite this definition using quantifiesrs:

$$\forall\varepsilon((\varepsilon > 0) \Rightarrow \exists\delta(\delta > 0 \wedge (0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon)))$$

(where ε and δ are real numbers). This can also be written a bit shorter as:

$$\forall\varepsilon > 0\ \exists\delta > 0\ (0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon).$$

6. The definition is as follows: the sequence $a_1, a_2, \ldots$ converges to a number L if for any positive number ε there exists an (integer) index N such that for any $n \geq N$, $|a_n - L| < \varepsilon$. We rewrite this definition using quantifiers:

$$\exists L \forall \varepsilon ((\varepsilon > 0) \Rightarrow (\exists N \forall n ((n \geq N) \to (|a_n - L| < \varepsilon))))$$

(where L and ε are real numbers, and n and N are natural numbers). This can also be expressed as:

$$\exists L \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ |a_n - L| < \varepsilon.$$