Interior and complement problem.

Construction. Consider a topological space (X, τ) and a subset $A \subseteq X$. Form the following sequences of subsets, alternating taking interior and complement:

$$A, \quad int(A), \quad \overline{int(A)}, \quad int(\overline{int(A)}), \quad \overline{int(\overline{int(A)})}, \ldots,$$

$$\overline{A}, \quad int(\overline{A}), \quad \overline{int(\overline{A})}, \quad int(\overline{int(\overline{A})}), \ldots.$$

Question. Is it possible to get infinitely many different sets in each of these sequences? If not, what is the maximum number of different sets possible?

Example 1. Let the topological space be $\mathbb{R}$ with the usual topology, and $A = (0, 1)$. Then we get:

$A = (0, 1)$,

$int(A) = (0, 1)$,

$\overline{int(A)} = (-\infty, 0] \cup [1, \infty)$,

$int(\overline{int(A)}) = (-\infty, 0) \cup (1, \infty)$,

$\overline{int(\overline{int(A)})} = [0, 1]$,

after which the sets will start repeating as $int(\overline{int(\overline{int(A)})}) = (0, 1)$ again.

For the second sequence we get:

$\overline{A} = (-\infty, 0] \cup [1, \infty)$,

so we won't get any new sets here since the very first one already appeared in the first sequence. Thus we get a total of four different sets.

Example 2. Let the topological space be $\mathbb{R}$ with the usual topology, and $A = (0, 1]$. Then we get:

$A = (0, 1]$,

$int(A) = (0, 1)$,

$\overline{int(A)} = (-\infty, 0] \cup [1, \infty)$,

$int(\overline{int(A)}) = (-\infty, 0) \cup (1, \infty)$,

$\overline{int(\overline{int(A)})} = [0, 1]$, after which the sets will start repeating as $int(\overline{int(\overline{int(A)})}) = (0,1)$ again.

For the second sequence we get:

$\overline{A} = (-\infty, 0] \cup (1, \infty)$,

$int(\overline{A}) = (-\infty, 0) \cup (1, \infty)$,

which is the same as $int(\overline{int(A)})$ in the first sequence. So we get five different sets in the first sequence and just one new set in the second sequence, thus a total of six different sets.

Example 3. Let the topological space be $\mathbb{R}$ with the usual topology, and $A = (0,1] \cup \{2\}$. Then we get:

$A = (0,1] \cup \{2\}$,

$int(A) = (0,1)$,

$\overline{int(A)} = (-\infty, 0] \cup [1, \infty)$,

$int(\overline{int(A)}) = (-\infty, 0) \cup (1, \infty)$,

$\overline{int(\overline{int(A)})} = [0,1]$,

after which the sets will start repeating as $int(\overline{int(\overline{int(A)})}) = (0,1)$ again.

For the second sequence we get:

$\overline{A} = (-\infty, 0] \cup (1,2) \cup (2, \infty)$,

$int(\overline{A}) = (-\infty, 0) \cup (1,2) \cup (2, \infty)$,

$\overline{int(\overline{A})} = [0,1] \cup \{2\}$,

$\int(\overline{int(\overline{A})}) = (0,1)$,

which is the same as $int(A)$ in the first sequence. So we get five different sets in the first sequence and three new different sets in the second sequence, thus a total of eight different sets.

Example 4. $\mathbb{R}$ with the usual topology, and $A = (0,1) \cup (1,2] \cup \{3\}$

This example was worked out in class and gave a total of 10 different sets in the two sequences.

Example 5. $\mathbb{R}$ with the usual topology, and $A = \mathbb{Q}$ (the set of rational numbers).

Then we get:

$A = \mathbb{Q}$,

$int(A) = \emptyset$,

$\overline{int(A)} = \mathbb{R}$,

whose interior is $\mathbb{R}$ again, then the complement is $\emptyset$, interior is $\emptyset$ again, the complement is $\mathbb{R}$, and so on, so we get only three different sets in the first sequence.

For the second sequence we get:

$\overline{A} = \mathbb{R} - \mathbb{Q}$ (the set of irrational numbers),

$int(\overline{A}) = \emptyset$,

so again we get repetition of $\emptyset$ and $\mathbb{R}$. Thus we get four different sets total in the two sequences.

Answer to the question given above: it is not possible to get infinitely many sets. The maximum possible number of different sets is seven in each sequence, so 14 total. However, this is not easy to prove using the definitions of interior and complement (the proof is long and technical). We will see later how this can be easily proved using modal logics.

Exercise. Construct a subset of $\mathbb{R}$ for which you get 14 different sets in these two sequences.