Topological Spaces.

Def. Let X be a non-empty set. Then $\tau \subseteq \mathcal{P}(x)$ is a topology on X if it satisfies:

(1) $\emptyset \in \tau$,

(2) $X \in \tau$,

(3) If $A, B \in \tau$, then $A \cap B \in \tau$,

(4) If $A_i \in \tau$ for each $i \in I$, then $\cup A_i \in \tau$.

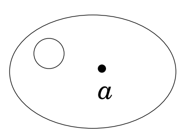
Remarks. (3) implies that the intersection of any finite number of elements of τ is also in τ, for example, if $A, B, C \in \tau$, then $A \cap B \in \tau$, and then $(A \cap B) \cap C \in \tau$, i.e. $A \cap B \cap C \in \tau$.

(4) means that in particular, the union of any finite number of elements of τ is in τ, for example, if $A, B, C \in \tau$, then $A \cup B \in \tau$ and $A \cup B \cup C \in \tau$.

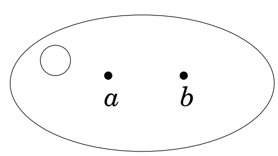
Def. Elements of τ are called open sets. The pair (X, τ) is called a topological space.

Examples.

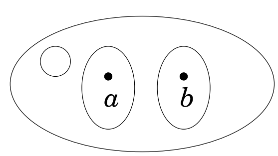
1. $X = \{a\}$, $\mathcal{P}(X) = \{\emptyset, X\}$. There is only one topology on X:
   $\tau = \{\emptyset, X\} = \mathcal{P}(X)$.



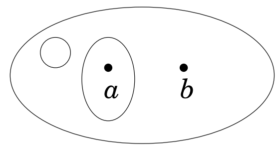
2. $X = \{a, b\}$, $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, X\}$. There are four different topologies on X.
   Topology $\tau_1 = \{\emptyset, X\}$.



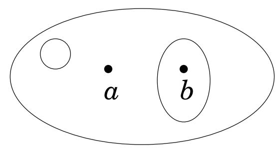
Topology $\tau_2 = \{\emptyset, \{a\}, \{b\}, X\} = \mathcal{P}(X)$.



Topology $\tau_3 = \{\emptyset, \{a\}, X\}$.



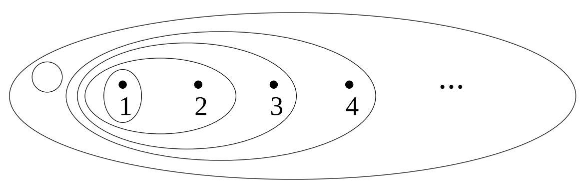
This space is called the Sierpinski space.
Topology $\tau_4 = \{\emptyset, \{b\}, X\}$.



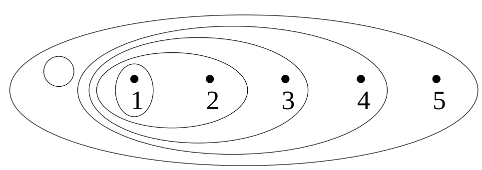
3. For any non-empty set X, consider $\tau = \{\emptyset, X\}$. Then τ is a topology, it is called the trivial (aka indiscrete) topology.

4. For any non-empty set X, consider $\tau = \mathcal{P}(X)$. Then τ is a topology, it is called the discrete topology.

5. $X = \mathbb{N}$, $\tau = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \mathbb{N}\}$. Then τ is a topology. Notice that in this example, for any $A, B \in \tau$, either $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$. If $B \subseteq A$, then $A \cap B = B$ and $A \cup B = A$.



6. $X = \{1, 2, 3, 4, 5\}$. (Remark: can do this for any finite $|X|$.) Consider $\tau = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, X\}$.



7. $X = \mathbb{R}$, the set of real numbers. The "usual" topology τ on $\mathbb{R}$ is defined as follows. A subset A of $\mathbb{R}$ is in τ if for every $x \in A$, there exists an open interval (a, b) containing x such that $(a, b) \subseteq A$.
For example, $A = (0, 1) \in \tau$, because for any point $x \in (0, 1)$, consider $(a, b) = (0, 1)$. We have $x \in (0, 1) \subseteq A$.
However, $A = [0, 1] \notin \tau$. For example, consider $x = 0$. There is no such open interval (a, b) that $0 \in (a, b)$ and $(a, b) \subseteq A$.

8. Let X be any non-empty set, and $p \in X$. Define τ as follows. A subset A of X is open if it is either empty or contains the point p. The point p is called the particular point, and this topology is called a particular point topology.

9. Let X be any non-empty set, and $e \in X$. Define τ as follows. A subset A of X is open if it is either equal to X or does not contain the point e. The point e is called the excluded point, and this topology is called an excluded point topology.

Remark. The Sierpinski space (topology τ_3 in example 2) is both a particular point topological space (with $p = a$) and an excluded point topological space (with $e = b$).

Def. If (X, τ) is a topological space and $A \subseteq X$, then A is called closed if $X - A$ is open.

Example. In $\mathbb{R}$ with the usual topology, closed intervals such as $[0, 1]$ are closed. Also, sets that are unions of closed intervals, e.g. $(-\infty, 0] \cup [1, 2] \cup [3, 4]$, are closed.

Properties. The union of any two closed sets is closed. The intersection of any number of closed sets is closed.

Def. If (X, τ) is a topological space and $A \subseteq X$. The interior of A, $int(A)$, is the largest open set contained in A. It is also the union of all open sets contained in A. The closure of A, $cl(A)$, is the smallest closed set containing A. It is also the intersection of all closed sets containing A.

Examples.

1. $X = \{a, b, c\}$, τ is particular point topology with $p = a$.
   The closed sets are: X, $\{b, c\}$, $\{c\}$, $\{b\}$, $\emptyset$.
   Then $cl(\{b, c\}) = \{b, c\}$, $cl(\{a, c\}) = X$.
2. $\mathbb{R}$, with the usual topology. Then $cl([0, 1]) = [0, 1]$, $cl((0, 1)) = [0, 1]$,
   $cl((-\infty, -3] \cup (2, 5] \cup [7, \infty)) = (-\infty, -3] \cup [2, 5] \cup [7, \infty)$.

Properties. For any topological space (X, τ) and any subsets $A, B \subseteq X$, we have

- $int(\emptyset) = \emptyset$
- $cl(\emptyset) = \emptyset$
- $int(X) = X$
- $cl(X) = X$
- $int(A) \subseteq A$
- $A \subseteq cl(A)$
- $int(A) \subseteq cl(A)$
- $int(int(A)) = int(A)$. Also, if B is open, then $int(B) = B$.
- $cl(cl(A)) = cl(A)$. Also, if B is closed, then $cl(B) = B$.

- $int(A \cap B) = int(A) \cap int(B)$ because
  1. $int(A) \cap int(B)$ is open
  2. $int(A) \cap int(B) \subseteq A \cap B$
  3. Suppose C is open and $C \subseteq A \cap B$, then $C \subseteq A$, $C \subseteq B$, Therefore $C \subseteq int(A)$, $C \subseteq int(B)$, thus $C \subseteq int(A) \cap int(B)$.
- $int(A) \cup int(B) \subseteq int(A \cup B)$.
  Here is an example when the above are not equal. Consider $\mathbb{R}$ with the usual topology, $A = (0, 1)$, $B = (-\infty, 0] \cup [1, \infty)$.
  Then $int(A) \cup int(B) = (0, 1) \cup (-\infty, 0) \cup (1, \infty) = (-\infty, 0) \cup (0, 1) \cup (1, \infty)$ and $int(A \cup B) = int(\mathbb{R}) = \mathbb{R}$.
- $cl(A \cup B) = cl(A) \cup cl(B)$
- $cl(A \cap B) \subseteq cl(A) \cap cl(B)$. Exercise: give an example of $A, B \in \mathbb{R}$ such that the above sets are not equal.