MATH 110 **Lecture notes – 11**

Paradoxes in Logic and Set Theory

1. **Existence of irrational numbers.**

   About 2.5 thousand years ago, Pythagoreans proved that $\sqrt{2}$ was irrational, which was controversial at the time. Until then it was believed that all numbers are rational, i.e. can be expressed as quotients of integers.

2. **Paradox of enumeration. Which set is bigger?**

   Before formal set theory was introduced, the notion of the size of a set had been problematic. It had been discussed by Italian mathematicians Galileo Galilei and Bernard Bolzano (17th-18th centruries), among others.
   Are there more as as many natural numbers as perfect squares (squares of natural numbers)?
   The answer is yes, because for every natural number n there is a square number n^2, and likewise the other way around.
   The answer is no, because the squares are a proper subset of the naturals: every square is a natural number but there are natural numbers, like 2, which are not squares of natural numbers.
   By defining the notion of the size of a set in terms of its cardinality, the issue can be settled, as there is a bijection between the two sets involved.
   Georg Cantor (late 19th-early 20th centuries) invented set theory. Cantor established the importance of one-to-one correspondence between the members of two sets, defined infinite and well-ordered sets, and proved that the real numbers are more numerous than the natural numbers. He introduced the concept of cardinality. "I see it but I don't believe," Cantor wrote to Dedekind after proving that the set of points of a square has the same cardinality as that of the points on just an edge of the square: the cardinality of the continuum.

3. **Ross-Littlewood paradox. Balls accumulating or all gone?**

   This paradox is also known as the balls and vase problem or the ping pong ball problem.
   You have an empty box and an ifnite supply of balls, numbered by natural numbers, 1, 2, 3, etc. In the first step, you put balls 1 and 2 into the box, and take ball 1 out immediately. In the second step, you put balls 3 and 4 in, and take ball 2 out. Next, you put balls 5 and 6 in, and take ball 3 out. And so on,

each time you put two balls in, and take one out. The question is, how many balls will be in the box eventually?

On the one hand, the number of balls in the box certainly grows, so it looks like there will be infinitely many of them in the box. On the other hand, every single ball will eventually be removed, so eventually no balls will be in the box. You could ask, what does "eventually" mean, as it seems that we will never be done. To that, there is a simple mathematical argument, however, not accepted by all philosophers. Namely, let us decide to do all of the steps by noon. We will do the first step 1 hour before noon (i.e. at 11AM), the second step $\frac{1}{2}$ hours before noon (so, at 11:30AM), the third step $\frac{1}{4}$ before noon, etc. All of the steps will be performed by noon. How many balls will be in the box at noon?

Also, to make the result even more surprizing, imagine your friend also has a box with an infinite supply of balls, numbered the same way as yours, but his actions are a little bit different. In his first step, he puts balls 1 and 2 into the box, and takes ball 1 out immediately, just like you. In his second step, he puts balls 3 and 4 into the box, but takes ball 3 out (unlike you, who took ball 2 out). Next, he puts balls 5 and 6 in, and takes 5 out. And so on. How many balls will be in his box at noon?

It is easy to see that he will take out all odd-numbered balls and will leave all even-numbered balls in the box. So, at noon, his box will contain infinitely many balls. But isn't it surprising that each of you, while performing all steps simultaneously, puts 2 balls in and take 1 ball out, yet eventually you have different numbers of balls in your two boxes?

There is no general concensus on this paradox. Mathematicians say that the number of balls in the box depends on the conditions, while some pholosophers say the problem is underspecified, and some say the problem is ill-formed.

The number of balls that one ends up with depends on which of the balls are removed from the vase. In fact, we can remove the balls so that any number of them will be in the vase at noon.

Although the state of the balls and the vase is well-defined at every moment in time prior to noon, no conclusion can be made about any moment in time at or after noon. Thus, for all we know, at noon, maybe the vase just magically disappears, or something else happens to it. But we don't know, as the problem statement says nothing about this. Hence, like the previous solution, this solution states that the problem is underspecified, but in a different way than the previous solution.

The problem is ill-posed. To be precise, according to the problem statement, an infinite number of operations will be performed before noon, and then asks about the state of affairs at noon. But, as in Zeno's paradoxes, if infinitely many operations have to take place (sequentially) before noon, then noon is a point in time that can never be reached. On the other hand, to ask how many balls will be left at noon is to assume that noon will be reached. Hence there is a contradiction implicit in the very statement of the problem, and this contradiction is the assumption that one can somehow "complete" an infinite number of steps.

4. **Interesting number paradox and Berry paradox.**

The interesting number paradox arises from the attempt to classify natural numbers as "interesting" or "dull." The paradox states that all natural numbers are interesting. The "proof" is by contradiction: if the set of uninteresting numbers is nonempty, there would be a smallest uninteresting number but the smallest uninteresting number is itself interesting because it is the smallest uninteresting number, producing a contradiction.
Attempting to classify all numbers this way leads to a paradox of definition. Any hypothetical partition of natural numbers into interesting and dull sets seems to fail, since the definition of interesting is usually a subjective, intuitive notion of "interesting."
However, "interesting" can be "defined" objectively: for example, any number that can be described, in English, by a sentence of fewer than 12 words.
Consider "the smallest positive integer not definable in fewer than twelve words."
Another (more precise) variation: use sixty letters in place of twelve words.
Since there are only twenty-six letters, there are finitely many phrases of under sixty letters, and hence finitely many positive integers that are defined by phrases of under sixty letters. Since there are infinitely many positive integers, this means that there are positive integers that cannot be defined by phrases of under sixty letters. If there are positive integers that satisfy a given property, then there is a smallest positive integer that satisfies that property; therefore, there is a smallest positive integer satisfying the property "not definable in under sixty letters". This is the integer to which the above expression refers. The above expression is only fifty-seven letters long, therefore it is definable in under sixty letters, and is not the smallest positive integer not definable in under sixty letters, and is not defined by this expression. This is a paradox: there must be an integer defined by this expression, but since the expression is self-contradictory (any integer it defines is definable in under sixty letters), there cannot be any integer defined by it.

5. **Grelling-Nelson paradox. Autological and Heterological words.**

Early 20th century, German mathematicians Grelling and Nelson.
Suppose one interprets the adjectives "autological" and "heterological" as follows:

An adjective is autological (aka homological) if and only if it describes itself. For example, the English word "English" is autological, as are "short," "unhyphenated" and "pentasyllabic."

An adjective is heterological if it does not describe itself. Hence "long" is a heterological word (because it is not a long word), as are "hyphenated" and "monosyllabic."

Is "heterological" a heterological word? Whether we say yes or no, we get a contradiction.

6. **Russell's paradox. The set of all sets.**

This paradox, discovered by Bertrand Russell in 1901, showed that some early attempts of formalization of the naive set theory created by Georg Cantor led to a contradiction.

Naive set theory is one of several theories of sets used in the discussion of the foundations of mathematics. Unlike axiomatic set theories, which are defined using a formal logic, naive set theory is defined informally, in natural language. It describes the aspects of mathematical sets familiar in discrete mathematics, and suffices for the everyday usage of set theory concepts in contemporary mathematics.

Sets are of great importance in mathematics; in modern formal treatments, most mathematical objects (numbers, functions, etc.) are defined in terms of sets. Naive set theory suffices for many purpose, while also serving as a stepping-stone towards more formal treatments.

According to naive set theory, any definable collection is a set. Let R be the set of all sets that are not elements of themselves. If R is not an element of itself, then its definition dictates that it must contain itself, and if it contains itself, then it contradicts its own definition as the set of all sets that are not elements of themselves.

Remark. The Grelling-Nelson paradox (discussed above) can be translated into Bertrand Russell's famous paradox in the following way. First one must identify each adjective with the set of objects to which that adjective applies. So, for example, the adjective "red" is equated with the set of all red objects. Similarly, the adjective "pronounceable" is equated with the set of all pronounceable things, one of which is the word "pronounceable" itself. Thus, an autological word is understood as a set, one of whose elements is the set itself. The question of whether the word "heterological" is heterological becomes the question of whether the set of all sets not containing themselves contains itself as an element.

The modern axiomatic (formal) set theory does not assume that, for every property, there is a set of all things satisfying that property. In particular, even the set of all sets does not exist.

7. **Barber paradox - another variation of Russel's paradox.**

   The barber is the "one who shaves all those, and those only, who do not shave themselves." The question is, does the barber shave himself?
   Despite its popular name, however, the barber paradox is not really a paradox in the true sense of this word. A man who shaves exactly those men who do not shave themselves simply cannot and does not exist, and there are virtually no reasons to expect the opposite. This is in contrast with the set of all sets that do not contain themselves (from Russell's paradox), whose existence cannot be so easily dismissed as it follows from the very intuitive and widely relied upon axioms of naive set theory.
   The Barber paradox can also be thought of as "an impossible instruction."

8. **Crocodile paradox.**

   This paradox, coming from ancient Greece, is another example of an impossible instruction.
   The paradox states that a crocodile, who has stolen a child, promises the father that his child will be returned if and only if he correctly predicts what the crocodile will do.
   The transaction is logically smooth but unpredictable if the father guesses that the child will be returned, but a dilemma arises for the crocodile if the father guesses that the child will not be returned. In the case that the crocodile decides to keep the child, he violates his terms: the fathers's prediction has been validated, and the child should be returned. However, in the case that the crocodile decides to give back the child, he still violates his terms: the father's prediction has been falsified, and the child should not be returned. The question of what the crocodile should do is therefore paradoxical, and there is no justifiable solution. In other words, the "instruction" to the crocodile is sometimes impossible to follow.

9. **Newcomb's paradox. Omega and his boxes with money.**

   Omega is a hypothetical super-intelligent being used in some philosophical problems. Omega is most commonly used as the predictor in Newcomb's problem. In its role as predictor, Omega's predictions occur almost certainly. In some thought experiments, Omega is also taken to be super-powerful.
   Omega shows you two boxes, A and B, and offers you the choice of taking only box A, or both boxes A and B. Omega has put $1,000 in box B. If Omega thinks you will take box A only, he has put $1,000,000 in it. Otherwise he has left it empty. Omega has played this game many times, and has never been wrong in his predictions about whether someone will take both boxes or not.

What makes this a “paradox” is that it brings into sharp conflict two distinct intuitions we have about decision-making, which rarely bear on the same situation but clash in the case of Newcomb’s. The first intution is considering rational expectations, act so as to bring about desired outcomes. This suggests taking one box: we expect, based on the evidence, that we will find box A empty if we take two boxes. The second intuition is only act if your action will alter the outcome. This suggests taking two boxes: our decision to take one box or both cannot alter the outcome.

This paradox was created by Newcomb in 1960, first analyzed and published in a philosophy paper spread to the philosophical community by Nozick in 1969, and appeared in Martin Gardner’s Scientific American column in 1974. Today it is a much debated problem in the philosophical branch of decision theory.

In his 1969 article, Nozick wrote that “To almost everyone, it is perfectly clear and obvious what should be done. The difficulty is that these people seem to divide almost evenly on the problem, with large numbers thinking that the opposing half is just being silly.”

Game theory offers two strategies for this game that rely on different principles: the expected utility principle and the strategic dominance principle. The problem is called a paradox because two analyses that both sound intuitively logical give conflicting answers to the question of what choice maximizes the player’s payout.

Considering the expected utility when the probability of the predictor being right is almost certain or certain, the player should choose box A. This choice statistically maximizes the player’s winnings, setting them at about $1,000,000 per game.

Under the dominance principle, the player should choose the strategy that is always better; choosing both boxes A and B will always yield $1,000 more than only choosing A. However, the expected utility of “always $1,000 more than A” depends on the statistical payout of the game; when Omega’s prediction is almost certain or certain, choosing both A and B sets player’s winnings at about $1,000 per game.

10. **Unexpected hanging.**

A judge tells a condemned prisoner that he will be hanged at noon on one weekday in the following week but that the execution will be a surprise to the prisoner. He will not know the day of the hanging until the executioner knocks on his cell door at noon that day.

Having reflected on his sentence, the prisoner draws the conclusion that he will escape from the hanging. He begins by concluding that the “surprise hanging” can’t be on Friday, as if he hasn’t been hanged by Thursday, there is only one day left - and so it won’t be a surprise if he’s hanged on Friday. Since the judge's sentence stipulated that the hanging would be a surprise to him, he concludes it cannot occur on Friday.

He then reasons that the surprise hanging cannot be on Thursday either, because Friday has already been eliminated and if he hasn't been hanged by Wednesday noon, the hanging must occur on Thursday, making a Thursday hanging not a surprise either. By similar reasoning he concludes that the hanging can also not occur on Wednesday, Tuesday or Monday. Joyfully he retires to his cell confident that the hanging will not occur at all.

The next week, the executioner knocks on the prisoner's door at noon on Wednesday which, despite all the above, was an utter surprise to him. What the judge said came true.

Other versions of the paradox replace the death sentence with a surprise fire drill, examination, pop quiz, or a lion behind a door.

Despite significant academic interest, there is no consensus on its precise nature and consequently a final correct resolution has not yet been established. One approach, logical analysis, suggests that the problem arises in a self-contradictory self-referencing statement at the heart of the judge's sentence. Epistemological studies of the paradox have suggested that it turns on our concept of knowledge. Even though it is apparently simple, the paradox's underlying complexities have even led to its being called a "significant problem" for philosophy.

The informal nature of everyday language allows for multiple interpretations of the paradox. In the extreme case, a prisoner who is paranoid might feel certain in his knowledge that the executioner will arrive at noon on Monday, then certain that he will come on Tuesday and so forth, thus ensuring that every day he is not hanged really is a "surprise" to him, but that the day of his hanging he was indeed expecting to be hanged.

Formulation of the judge's announcement into formal logic is made difficult by the vague meaning of the word "surprise." An attempt at formulation might be:

(A) The prisoner will be hanged next week and the date of the hanging will not be deducible the night before from the assumption that the hanging will occur during the week.

Given this announcement the prisoner can deduce that the hanging will not occur on the last day of the week. However, in order to reproduce the next stage of the argument, which eliminates Thursday, the prisoner must argue that his ability to deduce, from statement (A), that the hanging will not occur on the last day, implies that a last-day hanging would not be surprising. But since the meaning of "surprising" has been restricted to not deducible from the assumption that the hanging will occur during the week instead of not deducible from statement (A), the argument is blocked.

This suggests that a better formulation would be:

(B) The prisoner will be hanged next week and its date will not be deducible the night before using this statement as an axiom.

The first objection often raised to the logical school's approach is that it fails to explain how the judge's announcement appears to be vindicated after the fact. If the judge's statement is self-contradictory, how does he manage to be right all along? This objection rests on an understanding of the conclusion to be that the judge's statement is self-contradictory and therefore the source of the paradox. However, the conclusion is more precisely that in order for the prisoner to carry out his argument that the judge's sentence cannot be fulfilled, he must interpret the judge's announcement as (B). A reasonable assumption would be that the judge did not intend (B) but that the prisoner misinterprets his words to reach his paradoxical conclusion. The judge's sentence appears to be vindicated afterwards but the statement which is actually shown to be true is that "the prisoner will be psychologically surprised by the hanging". This statement in formal logic would not allow the prisoner's argument to be carried out.

A related objection is that the paradox only occurs because the judge tells the prisoner his sentence (rather than keeping it secret) which suggests that the act of declaring the sentence is important. Some have argued that since this action is missing from the logical school's approach, it must be an incomplete analysis.