Exclusive or.
Compound statements. Order of operations.
Logically equivalent statements.
Expressing some operations in terms of others.

Exclusive or (exclusive disjunction).

Exclusive or (aka exclusive disjunction) is an operation denoted by $\oplus$ and defined by the following truth table:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

So, $P \oplus Q$ means "either P or Q, but not both."

Compound statements.

A compound statement is an expression with one or more propositional variable(s) and two or more logical connectives (operators).

Examples:

- $(P \wedge Q) \to R$
- $(\neg P \to (Q \vee P)) \vee (\neg Q)$
- $\neg(\neg P)$.

Given the truth value of each propositional variable, we can determine the truth value of the compound statement.

Order of operations.

Just like in algebra, expressions in parentheses are evaluated first. Otherwise, negations are performed first, then conjunctions, followed by disjunctions (including exclusive disjunctions), implications, and, finally, the biconditionals.

For example, if P is True, Q is False, and R is True, then the truth value of $\neg P \to Q \wedge (R \vee P)$ is computed as follows: first, in parentheses we have $R \vee P$ is True, next, $\neg P$ is False, then $Q \wedge (R \vee P)$ is False, so $\neg P \to Q \wedge (R \vee P)$ is True.

In other words, $\neg P \to Q \wedge (R \vee P)$ is equivalent to $(\neg P) \to (Q \wedge (R \vee P))$.

Remark. The order given above is the most often used one. Some authors use other orders. In particular, many authors treat $\wedge$ and $\vee$ equally (so these are evaluated in the order in which they appear in the expression, just like $\times$ and $\div$ in algebra), and $\to$ and $\leftrightarrow$ equally (but after $\wedge$ and $\vee$, like $+$ and $-$ are performed after $\times$ and $\div$). To avoid confusion in this class, we will use parentheses to make the order clear.

Logically equivalent statements.

Two compound statements are called logically equivalent if they always have the same truth values (for any combination of the truth values of their components, i.e. variables). In other words, they are logically equivalent when they have identical truth tables.

Example: $P \to Q$ and $\neg P \vee Q$ are logically equivalent. Indeed,

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Comparing the third and fifth columns of the above table, we see that the truth values of $P \to Q$ and $\neg P \vee Q$ are always the same.

Logical equivalence is denoted by the symbol $\equiv$, e.g. we write

$$P \to Q \equiv \neg P \vee Q.$$

Expressing some operations in terms of others.

From the six operations $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$, some operations can be expressed in terms of others. For example, as we saw above,

$P \rightarrow Q \equiv \neg P \vee Q$.

Also, it can be checked using the truth tables that

$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$,

$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$,

$P \oplus Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$,

$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Observations:

1. Any operation can be defined in terms of $\wedge$, $\vee$, and $\neg$.
2. Since $\wedge$ can be defined in terms of $\vee$ and $\neg$, any operation can be defined in terms of these two.
3. Since $\vee$ can be defined in terms of $\wedge$ and $\neg$, any operation can be defined in terms of these two as well.

Questions:

1. Can $\neg$ be defined in terms of $\wedge$ and $\vee$?
2. Can $\wedge$ and $\vee$ be defined in terms of $\rightarrow$ and $\neg$?
3. Can all of these six operations be expressed in terms of just one of them?
4. Does there exist any other operation (an operation can be defined by a truth table) that could be used to define all these 6?