MATH 110 **Homework 3**

Consider the operation $X \star Y = \neg(X \wedge Y)$ and the following axiom system (given in class). Recall from class (review lecture notes if needed) that all of the operations $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ can be expressed in terms of $\star$.

Axioms:

1. $(X \wedge Y) \rightarrow X$
2. $(X \wedge Y) \rightarrow Y$
3. $X \rightarrow (X \vee Y)$
4. $Y \rightarrow (X \vee Y)$
5. $(\neg\neg X) \rightarrow X$
6. $X \rightarrow (Y \rightarrow X)$
7. $X \rightarrow (Y \rightarrow (X \wedge Y))$
8. $((X \rightarrow Y) \wedge (X \rightarrow \neg Y)) \rightarrow \neg X$
9. $((X \rightarrow Z) \wedge (Y \rightarrow Z)) \rightarrow ((X \vee Y) \rightarrow Z)$
10. $((X \rightarrow Y) \wedge (X \rightarrow (Y \rightarrow Z))) \rightarrow (X \rightarrow Z)$

Rule of inference: (Modus Ponens) $\dfrac{X,\ X \rightarrow Y}{Y}$

Problems

1. Write axioms 1, 3, and 5 using only the operation $X \star Y$ defined above.
2. Derive $(X \vee Y) \rightarrow (Y \vee X)$ from the above axiom system.
   Hint: Use axiom 9, in which replace Z by $Y \vee X$. Observe that the right hand side is what you want to derive. To use Modus Ponens, you need to derive the left hand side. This can be done as follows. Take axiom 7 and replace each X in it with $X \rightarrow (Y \vee X)$ and replace each Y with $Y \rightarrow (Y \vee X)$. Observe that these two expressions are basically axioms 4 and 3, respectively, with some variable substitution (namely, variables X and Y are switched). Now, using this rewritten axiom 7 and Modus Ponens twice, obtain the left hand side of your rewritten axiom 9. Then, use Modus Ponens to obtain the right hand side of it.
3. Derive $X \rightarrow X$.
   Hint: The idea is similar to the previous problem. Use axiom 10 first, in which replace Y with $X \vee Y$ and replace Z with X. To obtain the left hand side of this formula, use axioms 3, 6, and 7 (axiom 7 is used to connect two pieces, just like in the previous problem).