2.7 Tautologies and Contradictions

The symbols $\sim$, $\vee$, $\wedge$, $\Rightarrow$, and $\Leftrightarrow$ are sometimes referred to as **logical connectives.** From given statements, we can use these logical connectives to form more intricate statements. For example, the statement $(P \vee Q) \wedge (P \vee R)$ is a statement formed from the given statements P, Q, and R and the logical connectives $\vee$ and $\wedge$. We call $(P \vee Q) \wedge (P \vee R)$ a compound statement. More generally, a **compound statement** is a statement composed of one or more given statements (called **component statements** in this context) and at least one logical connective. For example, for a given component statement P, its negation $\sim P$ is a compound statement.

The compound statement $P \vee (\sim P)$, whose truth table is given in Figure 2.8, has the feature that it is true regardless of the truth value of P.

A compound statement S is called a **tautology** if it is true for all possible combinations of truth values of the component statements that comprise S. Hence $P \vee (\sim P)$ is

P	$\sim P$	$P \vee (\sim P)$
T	F	$\boldsymbol{T}$
F	T	$\boldsymbol{T}$

Figure 2.8 An example of a tautology

P	Q	$\sim Q$	$P \Rightarrow Q$	$(\sim Q) \vee (P \Rightarrow Q)$
T	T	F	T	$\boldsymbol{T}$
T	F	T	F	$\boldsymbol{T}$
F	T	F	T	$\boldsymbol{T}$
F	F	T	T	$\boldsymbol{T}$

Figure 2.9 Another tautology

a tautology, as is $(\sim Q) \vee (P \Rightarrow Q)$. This latter fact is verified in the truth table shown in Figure 2.9.

Letting

$$P_1 : 3 \text{ is odd. and } P_2 : 57 \text{ is prime.}$$

we see that not only is

57 is not prime, or 57 is prime if 3 is odd.

a true statement, but $(\sim P_2) \vee (P_1 \Rightarrow P_2)$ is true regardless of which statements P_1 and P_2 are being considered.

On the other hand, a compound statement S is called a **contradiction** if it is false for all possible combinations of truth values of the component statements that are used to form S. The statement $P \wedge (\sim P)$ is a contradiction, as is shown in Figure 2.10. Hence the statement

3 is odd and 3 is not odd.

is false.

Another example of a contradiction is $(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$, which is verified in the truth table shown in Figure 2.11.

Indeed, if a compound statement S is a tautology, then its negation $\sim S$ is a contradiction.

P	$\sim P$	$P \wedge (\sim P)$
T	F	$\boldsymbol{F}$
F	T	$\boldsymbol{F}$

Figure 2.10 An example of a contradiction

P	Q	$\sim P$	$P \wedge Q$	$Q \Rightarrow \sim P$	$(P \wedge Q) \wedge (Q \Rightarrow \sim P)$
T	T	F	T	F	$\boldsymbol{F}$
T	F	F	F	T	$\boldsymbol{F}$
F	T	T	F	T	$\boldsymbol{F}$
F	F	T	F	T	$\boldsymbol{F}$

Figure 2.11 Another contradiction