4.4 Proofs Involving Sets

We now turn our attention to proofs concerning properties of sets. Recall, for sets A and B contained in some universal set U, that the **intersection** of A and B is

$$A \cap B = \{x \; : \; x \in A \text{ and } x \in B\},$$

the **union** of A and B is

$$A \cup B = \{x \; : \; x \in A \text{ or } x \in B\},$$

and the **difference** of A and B is

$$A - B = \{x \; : \; x \in A \text{ and } x \notin B\}.$$

The set $A - B$ is also called the **relative complement** of B in A, and the relative complement of A in U is called simply the **complement** of A and is denoted by $\overline{A}$. Thus, $\overline{A} = U - A$. In what follows, we will always assume that the sets under discussion are subsets of some universal set U.

Figure 4.1 shows Venn diagrams of $A - B$ and $A \cap \overline{B}$ for arbitrary sets A and B. The diagrams suggest that these two sets are equal. This is, in fact, the case. Recall that to show the equality of two sets C and D, we can verify the two set inclusions $C \subseteq D$

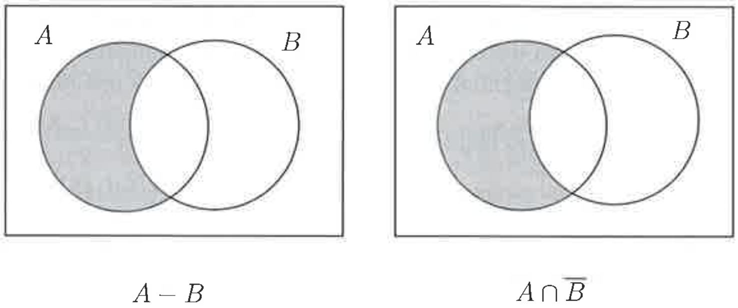


Figure 4.1 Venn diagrams for $A - B$ and $A \cap \overline{B}$

and $D \subseteq C$. To establish the inclusion $C \subseteq D$, we show that every element of C is also an element of D; that is, if $x \in C$ then $x \in D$. This is accomplished with a direct proof, by letting x be an (arbitrary) element of C and showing that x must belong to D as well. Recall that we need not be concerned if C contains no elements; for in this case $x \in C$ is false for every element x and so the implication "If $x \in C$, then $x \in D$." is true for all $x \in U$. As a consequence of this observation, if $C = \emptyset$, then C contains no elements and it follows that $C \subseteq D$.

Result 4.18 *For every two sets A and B,*

$$A - B = A \cap \overline{B}.$$

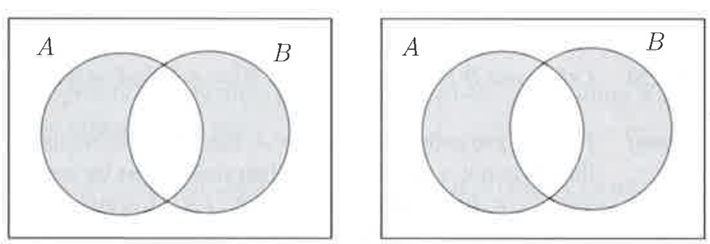
Proof First we show that $A - B \subseteq A \cap \overline{B}$. Let $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \notin B$, it follows that $x \in \overline{B}$. Therefore, $x \in A$ and $x \in \overline{B}$; so $x \in A \cap \overline{B}$. Hence $A - B \subseteq A \cap \overline{B}$.

Next we show that $A \cap \overline{B} \subseteq A - B$. Let $y \in A \cap \overline{B}$. Then $y \in A$ and $y \in \overline{B}$. Since $y \in \overline{B}$, we see that $y \notin B$. Now because $y \in A$ and $y \notin B$, we conclude that $y \in A - B$. Thus, $A \cap \overline{B} \subseteq A - B$. ■

PROOF ANALYSIS In the second paragraph of the proof of Result 4.18, we used y (rather than x) to denote an arbitrary element of $A \cap \overline{B}$. We did this only for variety. We could have used x twice. Once we decided to use distinct symbols, y was the logical choice since x was used in the first paragraph of the proof. This keeps our use of symbols consistent. Another possibility would have been to use a in the first paragraph and b in the second. This has some disadvantages, however. Since the sets are being called A and B, we might have a tendency to think that $a \in A$ and $b \in B$, which may confuse the reader. For this reason, we chose x and y over a and b.

Before leaving the proof of Result 4.18, we have one other remark. At one point in the second paragraph, we learned that $y \in A$ and $y \notin B$. From this we could have concluded (correctly) that $y \notin A \cap B$, but this is not what we wanted. Instead, we wrote that $y \in A - B$. It is always a good idea to keep our goal in sight. We wanted to show that $y \in A - B$; so it was important to keep in mind that it was the set $A - B$ in which we were interested, not $A \cap B$. ♦

Next, let's consider the Venn diagrams for $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$, which are shown in Figure 4.2. From these two diagrams, we might



$(A \cup B) - (A \cap B)$ $\qquad$ $(A - B) \cup (B - A)$

Figure 4.2 Venn diagrams for $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$

conclude (correctly) that the two sets $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$ are equal. Indeed, all that is lacking is a *proof* that these two sets are equal. That is, Venn diagrams can be useful in suggesting certain results concerning sets, but they are only drawings and do not constitute a proof.

Result 4.19 *For every two sets A and B,*

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

Proof First we show that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$. Let $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Since $x \in A \cup B$, it follows that $x \in A$ or $x \in B$. Without loss of generality, let $x \in A$. Since $x \notin A \cap B$, the element $x \notin B$. Therefore, $x \in A - B$ and so $x \in (A - B) \cup (B - A)$. Hence

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A).$$

Next we show that $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. Let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$, say the former. So $x \in A$ and $x \notin B$. Thus $x \in A \cup B$ and $x \notin A \cap B$. Consequently, $x \in (A \cup B) - (A \cap B)$. Therefore,

$$(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B),$$

as desired. ■

PROOF ANALYSIS In the proof of Result 4.19, when we were verifying the set inclusion

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A),$$

we concluded that $x \in A$ or $x \in B$. At that point, we could have divided the proof into two cases (*Case* 1. $x \in A$ and *Case* 2. $x \in B$); however, the proofs of the two cases would be identical, except that A and B would be interchanged. Therefore, we decided to consider only one of these. Since it really didn't matter which case we handled, we simply chose the case where $x \in A$. This was accomplished by writing:

Without loss of generality, assume that $x \in A$.

In the proof of the reverse set containment, we found ourselves in a similar situation, namely, $x \in A - B$ or $x \in B - A$. Again, these two situations were basically identical, and we simply chose to work with the first (former) situation. (Had we decided to assume that $x \in B - A$, we would have considered the *latter* case.) ♦

We now look at an example of a biconditional concerning sets.

Result 4.20 *Let A and B be sets. Then $A \cup B = A$ if and only if $B \subseteq A$.*

Proof First we prove that if $A \cup B = A$, then $B \subseteq A$. We use a proof by contrapositive. Assume that B is not a subset of A. Then there must be some element $x \in B$ such that $x \notin A$. Since $x \in B$, it follows that $x \in A \cup B$. However, since $x \notin A$, we have $A \cup B \neq A$.

Next we prove the converse, namely, if $B \subseteq A$, then $A \cup B = A$. We use a direct proof here. Assume that $B \subseteq A$. To verify that $A \cup B = A$, we show that $A \subseteq A \cup B$ and $A \cup B \subseteq A$. The set inclusion $A \subseteq A \cup B$ is immediate (if $x \in A$, then $x \in A \cup B$). It remains only to show then that $A \cup B \subseteq A$. Let $y \in A \cup B$. Thus $y \in A$ or $y \in B$. If

$y \in A$, then we already have the desired result. If $y \in B$, then since $B \subseteq A$, it follows that $y \in A$. Thus $A \cup B \subseteq A$. ■

PROOF ANALYSIS In the first paragraph of the proof of Result 4.20 we indicated that we were using a proof by contrapositive, while in the second paragraph we mentioned that we were using a direct proof. This really wasn't necessary as the assumptions we made would inform the reader what technique we were applying. Also, in the proof of Result 4.20, we used a proof by contrapositive for one implication and a direct proof for its converse. This wasn't necessary either. Indeed, it is quite possible to interchange the techniques we used (see Exercise 4.28). ♦

4.5 Fundamental Properties of Set Operations

Many results concerning sets follow from some very fundamental properties of sets, which, in turn, follow from corresponding results about logical statements that were described in Chapter 2. For example, we know that if P and Q are two statements, then $P \vee Q$ and $Q \vee P$ are logically equivalent. Similarly, if A and B are two sets, then $A \cup B = B \cup A$. We list some of the fundamental properties of set operations in the following theorem.

Theorem 4.21 *For sets A, B, and C,*

(1) *Commutative Laws*

(*a*) $A \cup B = B \cup A$
(*b*) $A \cap B = B \cap A$

(2) *Associative Laws*

(*a*) $A \cup (B \cup C) = (A \cup B) \cup C$
(*b*) $A \cap (B \cap C) = (A \cap B) \cap C$

(3) *Distributive Laws*

(*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) *De Morgan's Laws*

(*a*) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
(*b*) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

We present proofs of only three parts of Theorem 4.21, beginning with the commutative law of the union of two sets.

Proof of Theorem 4.21(1a) We show that $A \cup B \subseteq B \cup A$. Assume that $x \in A \cup B$. Then $x \in A$ or $x \in B$. Applying the commutative law for disjunction of statements, we conclude that $x \in B$ or $x \in A$; so $x \in B \cup A$. Thus, $A \cup B \subseteq B \cup A$. The proof of the reverse set inclusion $B \cup A \subseteq A \cup B$ is similar and is therefore omitted. ■

Next we verify one of the distributive laws.

Proof of Theorem 4.21(3a) First we show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Thus $x \in (A \cup B) \cap (A \cup C)$, as desired. On the other hand, if $x \in B \cap C$, then $x \in B$ and $x \in C$; and again, $x \in A \cup B$ and $x \in A \cup C$. So $x \in (A \cup B) \cap (A \cup C)$. Therefore, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To verify the reverse set inclusion, let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. So we may assume that $x \notin A$. Then the fact that $x \in A \cup B$ and $x \notin A$ implies that $x \in B$. By the same reasoning, $x \in C$. Therefore, $x \in B \cap C$, and so $x \in A \cup (B \cap C)$. Therefore, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. ■

As a final example, we prove one of De Morgan's laws.

Proof of Theorem 4.21(4a) First, we show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. Hence $x \notin A$ and $x \notin B$. Therefore, $x \in \overline{A}$ and $x \in \overline{B}$, so $x \in \overline{A} \cap \overline{B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Next we show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. Thus, $x \notin A$ and $x \notin B$, so $x \notin A \cup B$. Therefore, $x \in \overline{A \cup B}$. Hence $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. ■

PROOF ANALYSIS In the proof of the De Morgan law that we just presented, we arrived at the step $x \notin A \cup B$ at one point and then next wrote $x \notin A$ *and* $x \notin B$. Since $x \in A \cup B$ implies that $x \in A$ *or* $x \in B$, you might have expected us to write that $x \notin A$ *or* $x \notin B$ after writing $x \notin A \cup B$; but this would not be the correct conclusion. When we say that $x \notin A \cup B$, this is equivalent to writing $\sim(x \in A \cup B)$, which is logically equivalent to $\sim((x \in A)$ or $(x \in B))$. By the De Morgan law for the negation of the disjunction of two statements (or two open sentences), we have that $\sim((x \in A)$ or $(x \in B))$ is logically equivalent to $\sim(x \in A)$ and $\sim(x \in B)$; that is, $x \notin A$ *and* $x \notin B$. ♦

Proofs of some other parts of Theorem 4.21 are left as exercises.

4.6 Proofs Involving Cartesian Products of Sets

Recall that the **Cartesian product** (or simply the **product**) $A \times B$ of two sets A and B is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

Before looking at several examples of proofs concerning Cartesian products of sets, it is important to keep in mind that an arbitrary element of the Cartesian product $A \times B$ of two sets A and B is of the form (a, b), where $a \in A$ and $b \in B$.

Result 4.22 *Let A, B, C, and D be sets. If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.*

Proof Let $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. Since $A \subseteq C$ and $B \subseteq D$, it follows that $x \in C$ and $y \in D$. Hence $(x, y) \in C \times D$. ■

Result 4.23 *For sets A, B, and C,*

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof We first show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. Thus $y \in B$ or $y \in C$, say the former. Then $(x, y) \in A \times B$, and so $(x, y) \in (A \times B) \cup (A \times C)$. Consequently, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Next we show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$, say the former. Then $x \in A$ and $y \in B \subseteq B \cup C$. Hence $(x, y) \in A \times (B \cup C)$, implying that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. ■

We give one additional example of a proof involving the Cartesian products of sets.

Result 4.24 *For sets A, B, and C,*

$$A \times (B - C) = (A \times B) - (A \times C).$$

Proof First we show that $A \times (B - C) \subseteq (A \times B) - (A \times C)$. Let $(x, y) \in A \times (B - C)$. Then $x \in A$ and $y \in B - C$. Since $y \in B - C$, it follows that $y \in B$ and $y \notin C$. Because $x \in A$ and $y \in B$, we have $(x, y) \in A \times B$. Since $y \notin C$, however, $(x, y) \notin A \times C$. Therefore, $(x, y) \in (A \times B) - (A \times C)$. Hence $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

We now show that $(A \times B) - (A \times C) \subseteq A \times (B - C)$. Let $(x, y) \in (A \times B) - (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Since $(x, y) \in A \times B$, it follows that $x \in A$ and $y \in B$. Also, since $x \in A$ and $(x, y) \notin A \times C$, it follows that $y \notin C$. So $y \in B - C$. Thus $(x, y) \in A \times (B - C)$ and $(A \times B) - (A \times C) \subseteq A \times (B - C)$. ■

PROOF ANALYSIS We add one comment concerning the preceding proof. During the proof of $(A \times B) - (A \times C) \subseteq A \times (B - C)$, we needed to show that $y \notin C$. We learned that $(x, y) \notin A \times C$. However, this information alone did not allow us to conclude that $y \notin C$. Indeed, if $(x, y) \notin A \times C$, then $x \notin A$ *or* $y \notin C$. Since we knew, however, that $x \in A$ *and* $(x, y) \notin A \times C$, we were able to conclude that $y \notin C$. ♦

EXERCISES FOR CHAPTER 4

Section 4.1: Proofs Involving Divisibility of Integers

4.1. Let a and b be integers, where $a \neq 0$. Prove that if $a \mid b$, then $a^2 \mid b^2$.

4.2. Let $a, b \in \mathbf{Z}$, where $a \neq 0$ and $b \neq 0$. Prove that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

4.3. Let $m \in \mathbf{Z}$.

(a) Give a direct proof of the following: If $3 \mid m$, then $3 \mid m^2$.

(b) State the contrapositive of the implication in (a).

(c) Give a direct proof of the following: If $3 \nmid m$, then $3 \nmid m^2$.

(d) State the contrapositive of the implication in (c).

(e) State the conjunction of the implications in (a) and (c) using "if and only if".

4.4. Let $x, y \in \mathbf{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$, then $3 \mid (x^2 - y^2)$.

4.5. Let $a, b, c \in \mathbf{Z}$, where $a \neq 0$. Prove that if $a \nmid bc$, then $a \nmid b$ and $a \nmid c$.