7.2 Revisiting Quantified Statements

Many (in fact, most) of the statements we have encountered are quantified statements. Indeed, for an open sentence $P(x)$ over a domain S, we have often considered a quantified statement with a universal quantifier, namely

$$\forall x \in S, P(x): \text{ For every } x \in S, P(x). \text{ or If } x \in S, \text{ then } P(x).$$

or a quantified statement with an existential quantifier, namely

$$\exists x \in S, P(x): \text{ There exists } x \in S \text{ such that } P(x).$$

Recall that $\forall x \in S, P(x)$ is a true statement if $P(x)$ is true for every $x \in S$; while $\exists x \in S, P(x)$ is a true statement if $P(x)$ is true for at least one $x \in S$.

Example 7.1 *Let* $S = \{1, 3, 5, 7\}$ *and consider*

$$P(n): n^2 + n + 1 \textit{ is prime.}$$

for each $n \in S$*. Then both*

$$\forall n \in S,\ P(n):\ \textit{For every } n \in S,\ n^2+n+1 \textit{ is prime.}$$

and

$$\exists n \in S,\ P(n):\ \textit{There exists } n \in S \textit{ such that } n^2+n+1 \textit{ is prime.}$$

are quantified statements. Since

$$\begin{array}{lll} P(1): & 1^2+1+1=3 \textit{ is prime.} & \textit{is true,} \\ P(3): & 3^2+3+1=13 \textit{ is prime.} & \textit{is true,} \\ P(5): & 5^2+5+1=31 \textit{ is prime.} & \textit{is true,} \\ P(7): & 7^2+7+1=57 \textit{ is prime.} & \textit{is false,} \end{array}$$

it follows that $\forall n \in S, P(n)$ *is false and* $\exists n \in S, P(n)$ *is true. On the other hand, the statement*

$$Q:\ \textit{323 is prime.}$$

is not a quantified statement, but Q *is false (as* $323 = 17 \cdot 19$ *is not prime).* ♦

Let $P(x)$ be a statement for each x in some domain S. Recall that the negation of $\forall x \in S, P(x)$ is

$$\sim(\forall x \in S,\ P(x)) \equiv \exists x \in S,\ \sim P(x).$$

and the negation of $\exists x \in S, P(x)$ is

$$\sim(\exists x \in S,\ P(x)) \equiv \forall x \in S,\ \sim P(x).$$

Again, consider

$$P(n):\ n^2+n+1 \text{ is prime.}$$

from Example 7.1, which is a statement for each n in $S = \{1, 3, 5, 7\}$. The negation of $\forall n \in S, P(n)$ is

$$\exists n \in S, \sim P(n)\text{: There exists } n \in S \text{ such that } n^2+n+1 \text{ is not prime.}$$

is true as $7 \in S$ but $7^2+7+1 = 57$ is not prime. On the other hand, the negation of $\exists n \in S, P(n)$ is

$$\forall n \in S, \sim P(n)\text{: If } n \in S, \text{ then } n^2+n+1 \text{ is not prime.}$$

is false since, for example, $1 \in S$ and $1^2+1+1 = 3$ *is* prime.

In Chapter 2 we began a discussion of quantified statements containing two quantifiers. The following example concerns two quantifiers.

Example 7.2 *Consider*

$$P(s,t):\ 2^s+3^t \textit{ is prime.}$$

where s is a positive even integer and t is a positive odd integer. If we let S denote the set of positive even integers and T the set of positive odd integers, then the quantified statement

$$\exists s \in S, \exists t \in T, P(s, t)$$

can be expressed in words as

$$\text{There exist a positive even integer } s \text{ and a positive odd integer } t \text{ such that } 2^s + 3^t \text{ is prime.} \tag{7.1}$$

The statement (7.1) is true since

$$P(2, 1) : 2^2 + 3^1 = 7 \textit{ is prime.}$$

is true. On the other hand, the quantified statement

$$\forall s \in S, \forall t \in T, P(s, t)$$

can be expressed in words as

$$\text{For every positive even integer } s \text{ and every positive odd integer } t,\ 2^s + 3^t \text{ is prime.} \tag{7.2}$$

The statement (7.2) is false since

$$P(6, 3) : 2^6 + 3^3 = 91 \textit{ is a prime.}$$

is false, as $91 = 7 \cdot 13$ *is not a prime.* ♦

Let $P(s, t)$ be an open sentence, where the domain of the variable s is S and the domain of the variable t is T. Recall that the negations of the quantified statements $\exists s \in S, \exists t \in T, P(s, t)$ and $\forall s \in S, \forall t \in T, P(s, t)$ are

$$\sim(\exists s \in S, \exists t \in T, P(s, t)) \equiv \forall s \in S, \forall t \in T, \sim P(s, t)$$

and

$$\sim(\forall s \in S, \forall t \in T, P(s, t)) \equiv \exists s \in S, \exists t \in T, \sim P(s, t).$$

Therefore, the negation of the statement (7.1) is

For every positive even integer s and every positive odd integer t, $2^s + 3^t$ is not prime.

which is a false statement. On the other hand, the negation of the statement (7.2) is

There exist a positive even integer s and a
positive odd integer t such that $2^s + 3^t$ is not prime.

which is a true statement.

Quantified statements may also contain different kinds of quantifiers. For example, it follows by the definition of an even integer that for every even integer n, there exists an integer k such that $n = 2k$. There is another mathematical symbol with which you should be familiar. The symbol $\ni$ denotes the phrase *such that* (although some mathematicians simply write s.t. for "such that"). For example, let S denote the set of even integers again. Then

$$\forall n \in S,\ \exists k \in \mathbf{Z} \ni n = 2k \tag{7.3}$$

states:

For every even integer n, there exists an integer k such that $n = 2k$.

This statement can be reworded as:

If n is an even integer, then $n = 2k$ for some integer k.

If we interchange the two quantifiers in (7.3), we obtain, in words:

There exists an even integer n such that for every integer k, $n = 2k$.

This statement can also be reworded as

There exists an even integer n such that $n = 2k$ for every integer k.

This statement can be expressed in symbols as

$$\exists n \in S, \forall k \in \mathbf{Z}, n = 2k. \tag{7.4}$$

Certainly, the statements (7.3) and (7.4) say something totally different. Indeed, (7.3) is true and (7.4) is false.

Another such example of this is

For every real number x, there exists an integer n such that $|x - n| < 1$. **(7.5)**

This statement can also be expressed as

If x is a real number, then there exists an integer n such that $|x - n| < 1$.

In order to state (7.5) in symbols, let

$$P(x, n) : \ |x - n| < 1$$

where the domain of the variable x is $\mathbf{R}$ and the domain of the variable n is $\mathbf{Z}$. Thus (7.5) can be expressed in symbols as

$$\forall x \in \mathbf{R}, \exists n \in \mathbf{Z}, P(x, n).$$

The statement (7.5) is true, as we now verify.

Result 7.3 *For every real number x, there exists an integer n such that $|x - n| < 1$.*

Proof Let x be a real number. If we let $n = \lceil x \rceil$, where, recall, $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x, then $|x - n| = |x - \lceil x \rceil| = \lceil x \rceil - x < 1$. ■

Another example of a quantified statement containing two different quantifiers is

There exists a positive even integer m such that for every positive integer n, $\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{2}$. **(7.6)**

Let S denote the set of positive even integers and let

$$P(m, n) : \ \left|\tfrac{1}{m} - \tfrac{1}{n}\right| \leq \tfrac{1}{2}.$$

where the domain of the variable m is S and the domain of the variable n is $\mathbf{N}$. Thus, (7.6) can be expressed in symbols as

$$\exists m \in S, \forall n \in \mathbf{N}, P(m, n).$$

The truth of the statement (7.6) is now verifed.

Result 7.4 *There exists a positive even integer m such that for every positive integer n,*

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{2}.$$

Proof Consider $m = 2$. Let n be a positive integer. We consider three cases.

Case 1. $n = 1$. Then $\left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{1}{2} - \frac{1}{1}\right| = \frac{1}{2}$.

Case 2. $n = 2$. Then $\left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{1}{2} - \frac{1}{2}\right| = 0 < \frac{1}{2}$.

Case 3. $n \geq 3$. Then $\left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{1}{2} - \frac{1}{n}\right| = \frac{1}{2} - \frac{1}{n} < \frac{1}{2}$.

Thus $\left|\frac{1}{2} - \frac{1}{n}\right| \leq \frac{1}{2}$ for every $n \in \mathbf{N}$. ■

Let $P(s, t)$ be an open sentence, where the domain of the variable s is S and the domain of the variable t is T. The negation of the quantified statement $\forall s \in S, \exists t \in T, P(s, t)$ is

$$\begin{aligned}\sim(\forall s \in S, \exists t \in T, P(s, t)) &\equiv \exists s \in S, \sim(\exists t \in T, P(s, t)) \\ &\equiv \exists s \in S, \forall t \in T, \sim P(s, t);\end{aligned}$$

while the negation of the quantified statement $\exists s \in S, \forall t \in T, P(s, t)$ is

$$\begin{aligned}\sim(\exists s \in S, \forall t \in T, P(s, t)) &\equiv \forall s \in S, \sim(\forall t \in T, P(s, t)) \\ &\equiv \forall s \in S, \exists t \in T, \sim P(s, t).\end{aligned}$$

Consequently, the negation of the statement (7.5) is

There exists a real number x such that for every integer n, $|x - n| \geq 1$.

This statement is therefore false. The negation of the statement (7.6) is

For every positive even integer m, there exists a positive integer n such that $\left|\frac{1}{m} - \frac{1}{n}\right| > \frac{1}{2}$.

This too is false.

Let's consider the following statement, which has more than two quantifiers.

For every positive real number e, there exists a positive real number d such that for every real number x, $|x| < d$ implies that $|2x| < e$. **(7.7)**

If we let

$$P(x, d): \ |x| < d \quad \text{and} \quad Q(x, e): \ |2x| < e$$

where the domain of the variables e and d is $\mathbf{R}^+$ and the domain of the variable x is $\mathbf{R}$, then (7.7) can be expressed in symbols as

$$\forall e \in \mathbf{R}^+, \exists d \in \mathbf{R}^+, \forall x \in \mathbf{R}, P(x, d) \Rightarrow Q(x, e).$$

The statement (7.7) is in fact true, which we now verify.

Result 7.5 *For every positive real number e, there exists a positive real number d such that if x is a real number with $|x| < d$, then $|2x| < e$.*

Proof Let e be a positive real number. Now choose $d = e/2$. Let x be a real number with $|x| < d = e/2$. Then

$$|2x| = 2|x| < 2\left(\frac{e}{2}\right) = e,$$

as desired. ■

7.3 Testing Statements

We now turn our attention to the main topic of this chapter. For a given statement whose truth value is not provided to us, our task is to determine the truth or falseness of the statement and, in addition, show that our conclusion is correct by proving or disproving the statement, as appropriate.

Example 7.6 *Prove or disprove: There is a real number solution of the equation*

$$x^6 + 2x^2 + 1 = 0.$$

Strategy Observe that x^6 and x^2 are even powers of x. Thus if x is any real number, then $x^6 \geq 0$ and $x^2 \geq 0$, so $2x^2 \geq 0$. Adding 1 to $x^6 + 2x^2$ shows that $x^6 + 2x^2 + 1 \geq 1$. Hence it is impossible for $x^6 + 2x^2 + 1$ to be 0. These thoughts lead us to our solution. We begin by informing the reader that the statement is false, so the reader knows what we will be trying to do. ♦

Solution of Example 7.6 The statement is false. Let $x \in \mathbf{R}$. Since $x^6 \geq 0$ and $x^2 \geq 0$, it follows that $x^6 + 2x^2 + 1 \geq 1$ and so $x^6 + 2x^2 + 1 \neq 0$. ♦

For the preceding example, we wrote "Strategy" rather than "Proof Strategy" for two reasons: (1) Since the statement may be false, there may be no proof in this case. (2) We are essentially "thinking out loud", trying to convince ourselves whether the statement is true or false. Of course, if the statement turns out to be true, then our strategy may very well turn into a proof strategy.

Example 7.7 *Prove or disprove: Let $x, y, z \in \mathbf{Z}$. Then two of the integers x, y, and z are of the same parity.*

Strategy For any three given integers, either two are even or two are odd. So it certainly seems as if the statement is true. The only question appears to be whether what we said in the preceding sentence is convincing enough to all readers. We try another approach. ♦

Solution The statement is true.

Proof Consider x and y. If x and y are of the same parity, then the proof is complete. Thus we may assume that x and y are of opposite parity, say x is even and y is odd. If z is even, then x and z are of the same parity; while if z is odd, then y and z are of the same parity. ■

Of course, the preceding proof could have been done by cases as well.