Modal logic axioms valid in quotient spaces of finite CW-complexes and other families of topological spaces

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Outline

- Definitions
  - Topology
    - Particular point topological space
    - Excluded point topological space

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Particular point and excluded point topological spaces

Definition 1

Let X be any nonempty set and $p \in X$. The collection

$$T_p = \{S \subseteq X \mid p \in S \text{ or } S = \emptyset\}$$

of subsets of X is called the *particular point topology* on X.

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Definition 2

Let X be any nonempty set and $e \in X$. The collection

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of subsets of X is called the *excluded point topology* on X.

Quotient space of a CW-complex

Definition 3

Let X be a CW-complex. Its quotient space $Q(X)$ is a topological space whose points are in one-to-one correspondence with cells of X, and a subset of $Q(X)$ is open if and only if the union of the corresponding cells is open in X.

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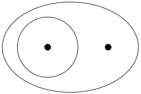
Example: The quotient space of the standard CW-complex of $\mathbb{R}P_1$ is the Sierpinski space.



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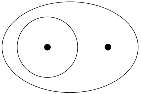

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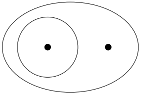

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If X is a CW-complex, its cell c is called a top cell if it is not in the boundary of any other cell.

A single point in $Q(X)$ is open if and only if it corresponds to a top cell.

Interpretation of $\mathcal{L}_\Box$ in topological spaces

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- $\mathcal{L}_\Box$ is the modal logic language consisting of propositional variables, $\wedge$, $\vee$, $\neg$, and $\Box$. Then, $\Diamond$ is defined by $\Diamond P = \neg\Box\neg P$.

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- A *topological model* of $\mathcal{L}_\Box$ is a pair $\langle X, \| \ \| \rangle$, where
  (1) X is a topological space, and
  (2) $\|\cdot\|$ is a function mapping formulas in $\mathcal{L}_\Box$ to subsets of X satisfying

$$\begin{aligned} \|F \wedge G\| &= \|F\| \cap \|G\| \\ \|F \vee G\| &= \|F\| \cup \|G\| \\ \|\neg F\| &= \|F\|^C = X \setminus \|F\| \\ \|\Box F\| &= \mathrm{int}(\|F\|) \end{aligned}$$

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- A formula F is called *valid* in a topological space X if for any topological model $\langle X, \| \ \| \rangle$, we have $\|F\| = X$.

Axioms

S4:

- All axioms of the classical propositional logic,
- Axiom K: $\Box(A \to B) \to (\Box A \to \Box B)$,
- Axiom T: $\Box A \to A$,
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M and G in particular point and excluded point topological spaces

Theorem 6

Both axioms M and G are valid in any particular point topological space.

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Theorem 7

(1) *Axiom M is valid in any excluded point topological space.*

(2) *Axiom G is valid in an excluded point topological space if and only if the space has only 1 or 2 points.*

M and G in quotient spaces of CW-complexes

Theorem 8

(1) *Axiom M is valid in the quotient space of any finite CW-complex.*

(2) *Axiom G is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.*

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Idea of proof:
(1) For any $c \in Q(X)$, $\exists t \in Q(X)$ corresponding to a top cell s.t. $c \in \mathrm{cl}(\{t\}) = \mathrm{cl}(\mathrm{int}(\{t\}))$.

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($\Rightarrow$) If a connected component contains more than one top cell, there are two top cells whose closures have non-empty intersection.

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Let $t_1, t_2 \in Q(X)$ correspond to such two top cells and let c correspond to a cell in the intersection of their closures.

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Let $t_1, t_2 \in Q(X)$ correspond to such two top cells and let c correspond to a cell in the intersection of their closures.
Consider a validation mapping such that $t_1 \in \|P\|$ and $t_2 \notin \|P\|$.

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Consider a validation mapping such that $t_1 \in \|P\|$ and $t_2 \notin \|P\|$.
Then $c \notin \|\Box\Diamond P\|$ and $c \notin \|\Box\Diamond\neg P\|$.

References

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Thank you!