This is not a coincidence! Peculiar patterns in some Calculus optimization problems explained

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Outline

Optimizing rectangle

Out of all rectangles with a given perimeter, which one has the greatest area?

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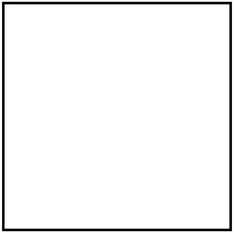

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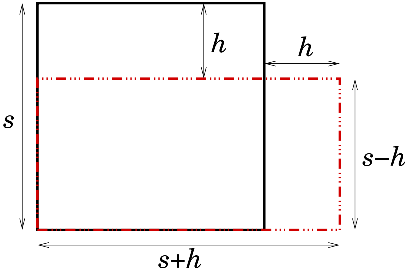

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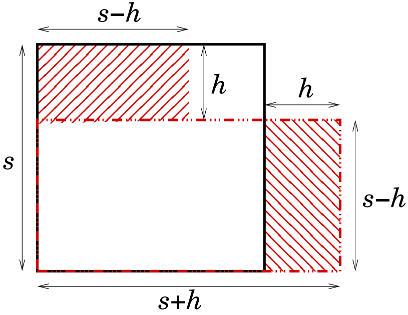

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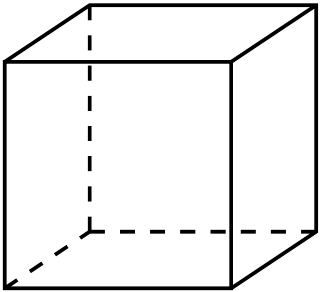

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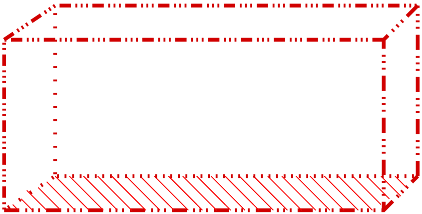

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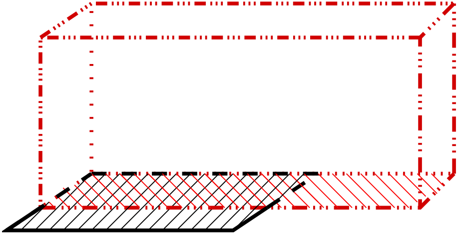

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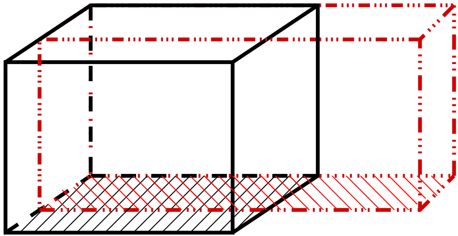

The rectangular field roblem

A farmer wants to fence off a rectangular field and divide it into 3 pens with fence parallel to one pair of sides. He has a total 2400 ft of fencing. What are the dimensions of the field that has the largest possible area?

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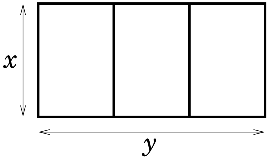

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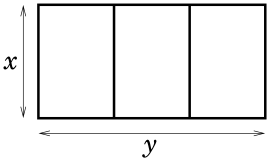

$y = \frac{2400-4x}{2} = 1200 - 2x$

$\text{Area}(x) = 1200x - 2x^2$

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$x = 300$ is an absolute maximum

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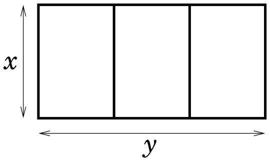

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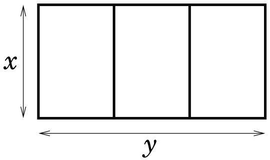

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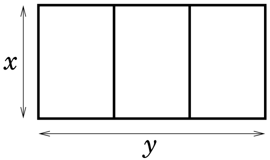
These are equal!

Why? Functional explanation



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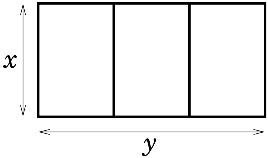

Let L be the total length of the vertical pieces.

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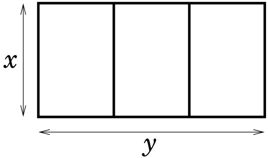

Let L be the total length of the vertical pieces.
$2400 - L$ is the total length of the horizontal pieces.

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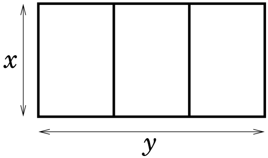

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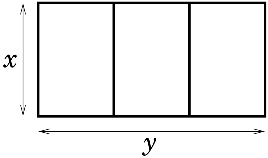

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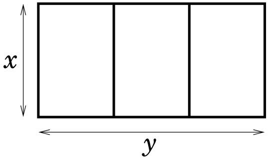

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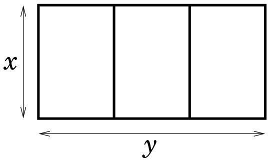

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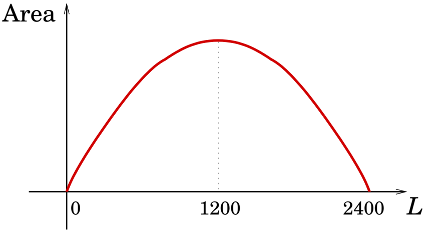

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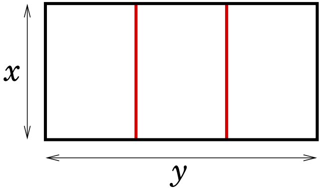

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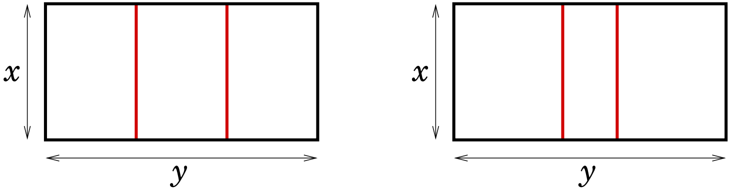

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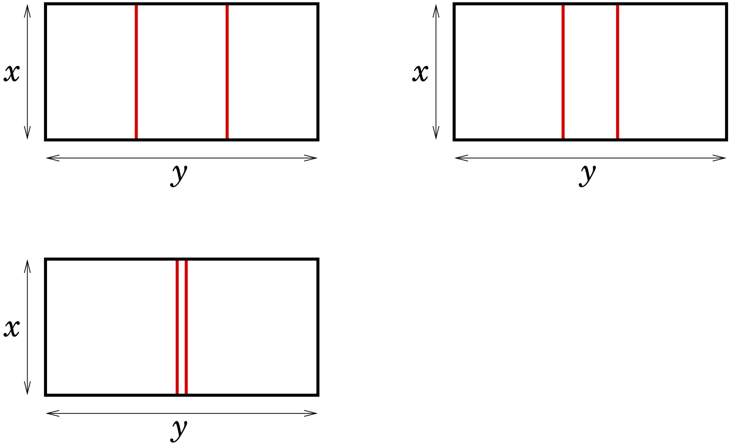

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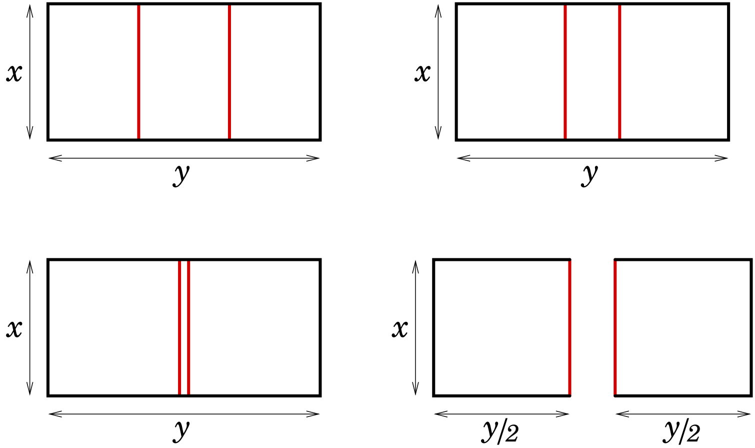

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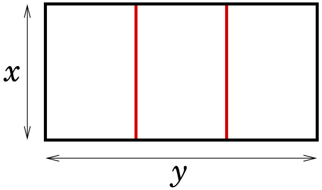

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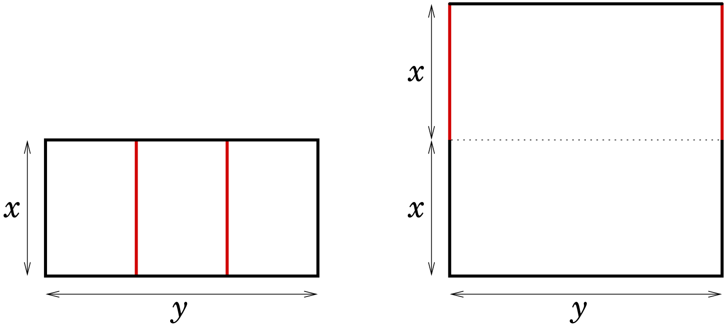

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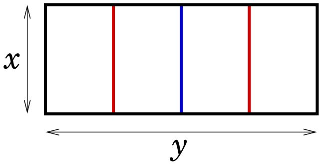

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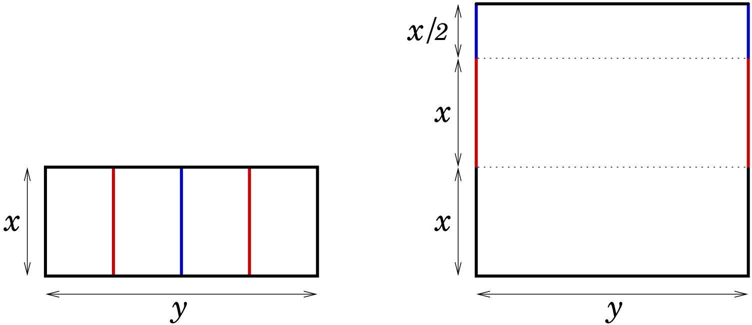

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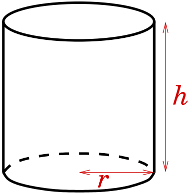

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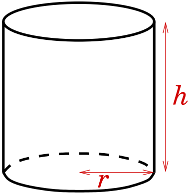

$$h = \frac{1000}{\pi r^2}$$
$$SA(r) = 2\pi r^2 + \frac{2000}{r}$$
$$SA'(r) = 4\pi r - \frac{2000}{r^2} = 0$$
$$r = \sqrt[3]{\frac{500}{\pi}} \text{ is an absolute minimum}$$
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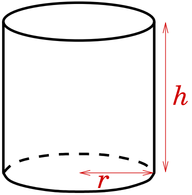

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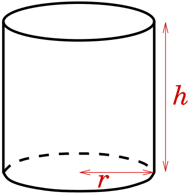

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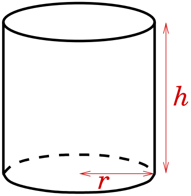
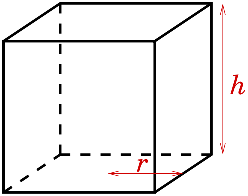
Observation: $d = 2\sqrt[3]{\frac{500}{\pi}}$ cm $\quad h = d!$

Why is this so?





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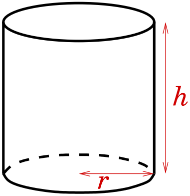
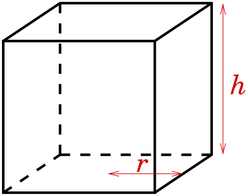


$V_{can} = A_{circle} h$



$V_{cube} = A_{square} h$

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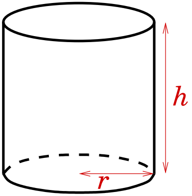
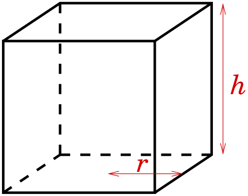
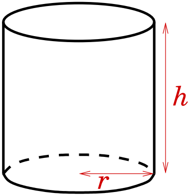


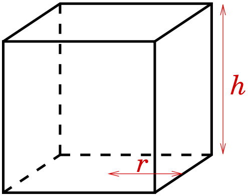


$$V_{can} = A_{circle} h \qquad V_{cube} = A_{square} h$$

$$V_{cube} = \frac{A_{square}}{A_{circle}} V_{can} = \frac{4r^2}{\pi r^2} V_{can} = \frac{4}{\pi} V_{can}$$

Why is this so?





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$SA_{can} = 2A_{circle} + P_{circle} h$

$SA_{cube} = 2A_{square} + P_{square} h$

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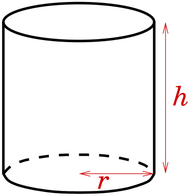

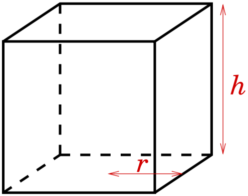

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$SA_{can} = 2A_{circle} + P_{circle}h$

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Question: is $\frac{P_{square}}{P_{circle}} = \frac{A_{square}}{A_{circle}}$?

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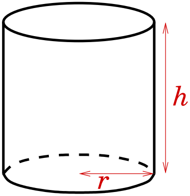

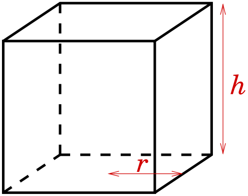

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Answer: $\frac{8r}{2\pi r} = \frac{4r^2}{\pi r^2}$

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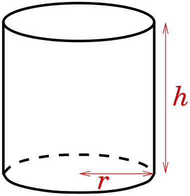

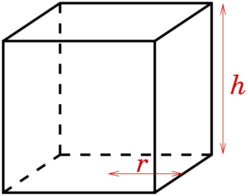

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Answer: $\frac{8r}{2\pi r} = \frac{4r^2}{\pi r^2}$ Yes!

Is that a coincidence?

Why $\frac{P_{square}}{P_{circle}} = \frac{A_{square}}{A_{circle}}$?

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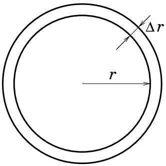

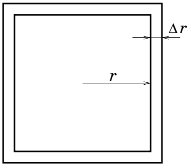

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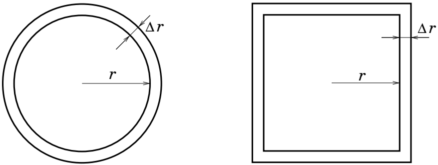

$$\frac{ar^2}{2ar} = \frac{br^2}{2br}$$

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Equivalently:

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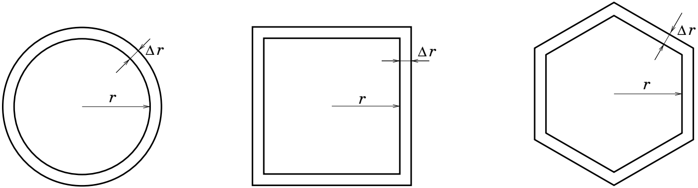
$$\frac{\pi r^2}{2\pi r} = \frac{4r^2}{8r}$$

The denominator is the derivative of the numerator!



$$\frac{ar^2}{2ar} = \frac{br^2}{2br}$$

Is that a coincidence?

Why $\frac{P_{square}}{P_{circle}} = \frac{A_{square}}{A_{circle}}$?

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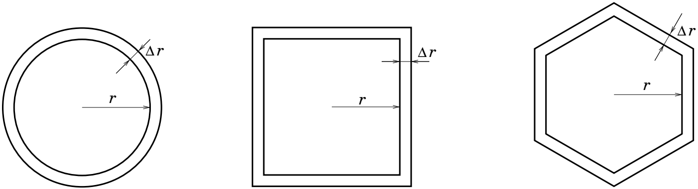

$$\frac{ar^2}{2ar} = \frac{br^2}{2br} = \frac{cr^2}{2cr}$$

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Why $\frac{P_{square}}{P_{circle}} = \frac{A_{square}}{A_{circle}}$?

Equivalently:

$$\frac{A_{circle}}{P_{circle}} = \frac{A_{square}}{P_{square}} = \frac{A_{hexagon}}{P_{hexagon}}$$

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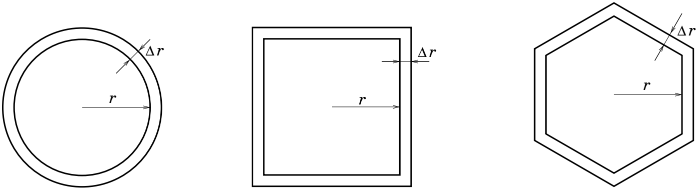

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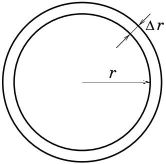

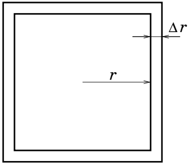

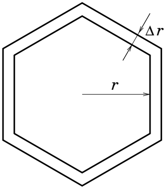

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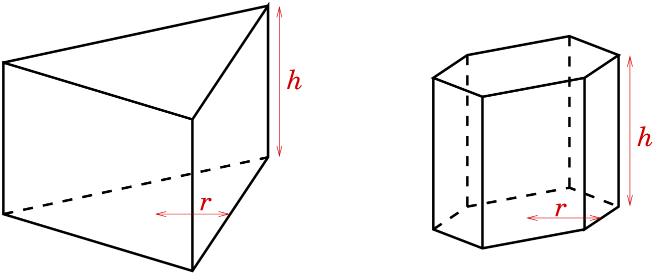
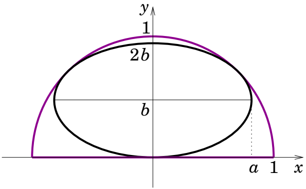
Other boxes



Optimal shape: $h = 2r$

The ellipse inscribed in a semi-circle problem

Of all ellipses inscribed in a semi-circle of radius 1, find the one with the largest possible area.
Hint: If the semicircle is given by the equation $x^2 + y^2 = 1$, $y \geq 0$, the ellipse should have equation of the form $\frac{x^2}{a^2} + \frac{(y-b)^2}{b^2} = 1$.



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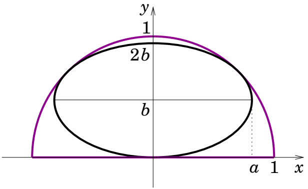

Answer: optimal dimensions are $a = \frac{\sqrt{6}}{3}$ and $b = \frac{\sqrt{2}}{3}$.

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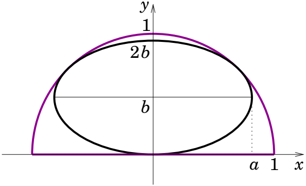

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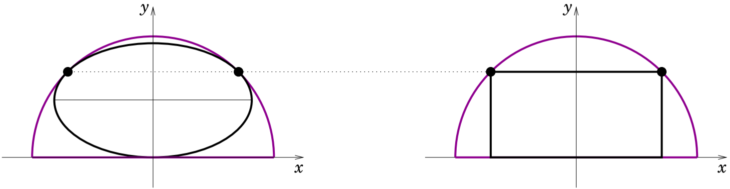

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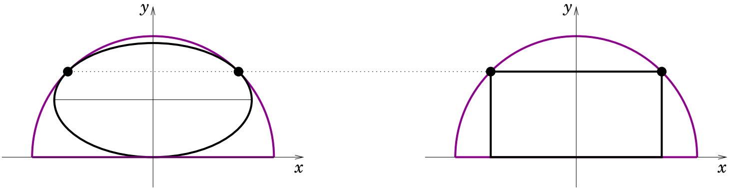

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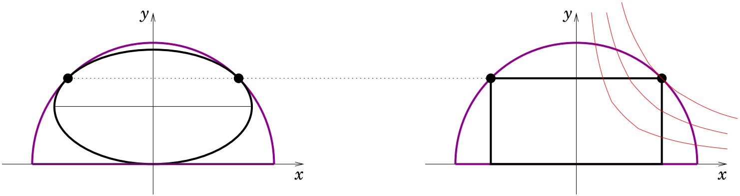

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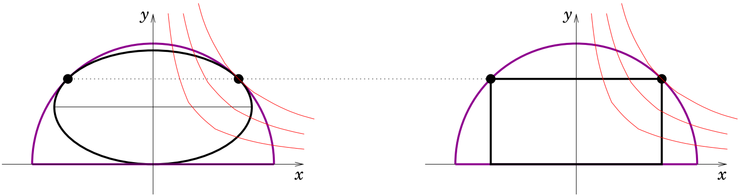

Thank you!